THROWING AWAY A BILLION DOLLARS:

The Cost of Suboptimal Exercise Strategies in the Swaptions Market

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ABSTRACT

This paper studies the costs of following single-factor exercise strategies for American swaptions when the term structure is actually driven by multiple factors. By their nature, single-factor models imply myopic exercise strategies since they assume perfectly correlated interest rate changes. Using a multi-factor string model of the term structure fitted to swaptions and caps data, we find that even the best single-factor exercise strategy can be very suboptimal. Based on ISDA estimates of swaption notional amounts outstanding, the total present value costs of following single-factor strategies could be as large as several billion dollars.

1. INTRODUCTION

Interest rate swaps have become one of the most important sectors in the global fixed income markets. The International Swaps and Derivatives Association (ISDA) estimates that the notional amount of interest rate swaps outstanding at the end of 1997 was \$22.3 trillion, more than four times the \$5.5 trillion notional of Treasury debt outstanding. Given the size of the swap market, it is not surprising that options to enter into swaps, or to cancel existing swaps, represent one of the most-widely-used classes of fixed income derivatives. Known as swap options or swaptions, the total notional amount of these derivatives outstanding at the end of 1997 was nearly \$1.7 trillion, more than ten times the \$15 billion notional of all Chicago Board of Trade Treasury note and bond futures options combined.

The importance of the swaptions market derives from the key role that swaptions play in corporate finance. Industry sources estimate that more than 50 percent of new agency and corporate debt issues are immediately swapped from fixed into floating or from floating into fixed. Debt issuers that swap their debt typically want the right to cancel the swap at future points in time. Similarly, debt issuers that do not immediately swap their debt often want the right to enter into a prespecified swap at later dates. Thus, swaptions arise as a natural outgrowth of efforts by debt issuers to preserve their flexibility throughout the financing cycle. Because of the long maturity of many debt issues, swaptions are often exercisable over horizons that span decades. Intuitively, swaptions can be viewed as calls or puts on coupon bonds, and our results are directly applicable to callable and puttable bonds. As with callable and puttable debt issues, a high percentage of swaptions have American-style exercise features.

Despite their importance, however, there are few areas in Finance where there is a larger divergence between theory and practice than for American-style swaptions. On one hand, there is an extensive and well established body of research showing that the term structure is driven by multiple factors.[1] On the other hand, most Wall Street firms exercise their American-style swaptions on the basis of simple single-factor term structure models. If term structure dynamics are determined by multiple factors, however, then exercise strategies implied by single-factor models may be far from optimal, resulting in significant erosion of the value of the swaption to the optionholder.

The reason for this suboptimality stems from the perfect correlation of points along the term structure inherent in single-factor models. Because of this perfect

[1] For example, see Brown and Dybvig (1986), Stambaugh (1988), Litterman and Scheinkman (1991), Longstaff and Schwartz (1992), Chen and Scott (1993), Gibbons and Ramaswamy (1993), Knez, Litterman, and Scheinkman (1994), Pearson and Sun (1994), Balduzzi, Das, Foresi, and Sundaram (1996), Dai and Singleton (1999), Piazzesi (2000), and Longstaff, Santa-Clara, and Schwartz (2000).

correlation, the swaption can be perfectly hedged with only the underlying swap and the decision to exercise depends myopically on only the current underlying swap value which becomes a sufficient statistic for term structure movements. In a multi-factor framework, however, the optimal exercise or stopping rule depends not only on the current value, but also on forward values of the underlying swap. By basing exercise decisions on only a subset of the relevant term structure information, single-factor stopping rules cannot generally be optimal.

Another way of seeing this is by noting that the optimal exercise boundary is a multi-dimensional surface when the term structure is driven by multiple factors. By its nature, however, a single-factor model forces the exercise boundary to be one-dimensional. In general, the multi-dimensional optimal exercise surface cannot be well approximated by the one-dimensional exercise boundary implied by a single-factor model. The same applies to the dynamic replication of the swaption. A single-factor hedging strategy cannot replicate an American swaption in a multi-factor framework. It is important to observe that this problem is inherent in the topology of single-factor models. No matter how well a single-factor model is parameterized to match the cross section of market option prices, it still implies perfect correlation and fails to capture the times series properties or dynamics of the term structure. The dynamics of the term structure, however, are fundamental in determining the value of American swaptions because of the intertemporal nature of the optimal stopping problem. The bottom line is that if a swaption holder uses a single-factor model to make exercise decisions, the present value A of cash flows realized will be less than or equal to the present value B of cash flows realized by following the optimal multi-factor exercise strategy.

While this latter point may seem obvious, it has been the source of some recent confusion among practitioners. The reason for this stems from an interesting apparent paradox. Specifically, if the holder of an American swaption uses a single-factor model to value the option as well as to make exercise decisions, the value implied by the single-factor model X may occasionally exceed the correct value B given by the multi-factor model.[2] Thus, the swaption holder could be led to believe that the swaption was actually more valuable using a single-factor model. The paradox is easily resolved, however, by recognizing that because the dynamics of the single-factor model are misspecified, the American swaption value X implied by the single-factor model need not equal the present value A of the actual cash flows obtained by following the single-factor exercise strategy in the real world. Hence, the American swaption value X implied by a misspecified single-factor term structure model is an illusionary number which can be a severely biased estimate of the present value A of actual cash flows realized from following the single-factor strategy. Thus, comparisons between X and B are meaningless.

[2]As examples, see the recent papers by Andersen and Andreasen (2000) and Brace and Womersley (2000).

We illustrate this point with several simple examples. In the first, an investor has a two-period American exchange option but mistakenly believes one of the two asset prices is constant. The investor calibrates a single-factor model to match exactly the market prices of European exchange options at each date. This calibrated single-factor model values the American exchange option at 1.37. It is easily shown, however, that following the single-factor strategy over time results in cash flows that have a present value of only 1.13, which is about 82 percent as much as the investor believes the option is worth. In the second example, we use a two-factor binomial tree to value options on a coupon bond. We then fit a time-inhomogeneous single-factor binomial tree to match the values of the European options throughout the tree. The calibrated single-factor tree implies that the value of an American option on the coupon bond is .514. Following the single-factor exercise strategy over time, however, results in cash flows that have a present value of .563, which is nearly ten percent higher than the value implied by the single-factor model. These two examples demonstrate that the American option value X implied by a misspecified single-factor model can be biased in either direction. This bias is particularly insidious since an investor using a single-factor model may not even be aware that the present value A of the cash flows he realizes does not equal the value X given by the model since he only observes ex post realizations.

This paper studies the costs of following single-factor exercise strategies for American-style swaptions in a realistic multi-factor term structure framework. In modeling interest rate dynamics, we use a multi-factor string model of the term structure similar to Santa-Clara and Sornette (2000), Longstaff and Schwartz (2000), and Longstaff, Santa-Clara, and Schwartz (2000). String models have the important advantages of being easily calibrated and providing a rich multi-factor description of the dynamic behavior of the term structure. Furthermore, since the string model specifies the dynamics of the entire curve, it is straightforward to accurately value American-style swaptions in this framework using the least squares Monte Carlo (LSM) technique of Longstaff and Schwartz.

Our approach consists of first fitting a time-homogeneous string model to sets of market prices for European swaptions and interest rate caps. Following Knez, Litterman, and Scheinkman (1994), Piazzesi (2000), Longstaff, Santa-Clara, and Schwartz (2000) and others who find evidence of four factors in term structure dynamics, we allow the correlation matrix of forward rates to be of rank four. We then simulate paths of the term structure using the fitted four-factor string model. Finally, applying the LSM algorithm to the simulated paths, we compare the values of American-style swaptions obtained using the best single-factor exercise strategy with those obtained using the optimal four-factor strategy.

The results have many important implications for swaptions markets. We show that even the best single-factor strategy can be very suboptimal. For many common American swaption structures, the present value loss from following a single-factor strategy rather than the optimal four-factor strategy can be as large as 10 to 20 cents per \$100 notional, or 5 to 20 percent of the American exercise premium. Based on ISDA estimates of swaption notional amounts outstanding, the total present value cost to swaptionholders of following single-factor exercise strategies could be as large as several billion dollars.

To provide insights into the properties of optimal exercise strategies in a multi-factor framework, we contrast them with the single-factor strategy. The optimal multi-factor strategy differs fundamentally from the single-factor strategy in three ways. First, the multi-factor strategy results in the swaption being exercised a higher percentage of the time. Second, the average amount realized when the swaption is exercised is greater under the multi-factor strategy than under the single-factor strategy. Third, the multi-factor strategy typically results in the swaption being exercised later than under the single-factor model, although earlier exercise also occurs frequently. Thus, there are no simple rules of thumb which could be used to improve the performance of a single-factor exercise strategy.

The remainder of this paper is organized as follows. Section 2 provides a brief introduction to the swaptions market. Section 3 presents the string model of the term structure. Section 4 describes the parameterization of the string model. Section 5 discusses the numerical approach and considers the effects of optimal and suboptimal exercise strategies on American swaption valuation. Section 6 presents the results about the costs of following suboptimal strategies. Section 7 summarizes the results and makes concluding remarks.

2. AN INTRODUCTION TO SWAPTIONS

This section provides a brief introduction to the swaptions market. We first consider how the market is structured. We then describe the various types of swaptions and their characteristics. Finally, we discuss the major valuation models used in practice.

2.1 The Swaptions Market.

The inception of the interest rate swap market in the early 1980s also led to the creation of the swaptions market. The dramatic growth in the swap market has resulted in swaptions becoming one of the most important fixed income derivative products. Swaptions play a unique role in fixed income markets since they are often exercisable over extended horizons; no other fixed income derivatives provide investors with such long-dated optionality.

Primary and secondary markets for swaps and swaptions are made by a world-wide network of swap dealers. Most of these dealers are members of ISDA which is an independent self-regulatory organization that has developed standards for documentation, contractual terms, dispute reconciliation, and other aspects of the swap business. Currently, there are nearly 200 primary members of ISDA. Swap dealers trade as principals by either taking the other side of customer transactions or by entering into swaps with other dealers. Since swaps and swaptions are contractual obligations rather than securities, a significant amount of legal infrastructure is needed before an entity can obtain swap lines and enter into swaps with dealers. To minimize the amount of legal work involved, many swap dealers trade only with counterparties with whom a standard ISDA master swap agreement has been negotiated. These master swap agreements apply to all of the swaps executed between the two counterparties and address such issues as the rights of offset in the event that one of the counterparties defaults on a specific swap.[3] The swap market is highly liquid and bid/ask spreads in the U.S. market are often a fraction of a basis point.

The dealer market is intermediated by a network of brokers who do not take positions for their own account but facilitate trading by providing a degree of anonymity for dealers wishing to trade with other dealers. This function is particularly complex because of the contractual nature of swaps and swaptions. For example, brokers face the challenge of arranging transactions between counterparties who are unknown to each other, but who may have credit exposure to the other counterparty once the transaction is executed. In addition, adverse selection problems are particularly acute since most active participants in these markets are large organizations with significant private information about future trading demands and client financing needs.[4]

2.2 Swap Characteristics.

The underlying instrument for a swaption is an interest rate swap. In a standard swap, two counterparties agree to exchange a stream of cash flows over some specified period of time. One counterparty receives a fixed annuity and pays the other a stream of floating cash flows tied to the three-month Libor rate. Counterparties are identified as either receiving fixed or paying fixed in the swap. Although principal is not exchanged at the end of a swap, it is often more intuitive to think of a swap as involving a mutual exchange of $1 at the end of the swap. From this perspective, the cash flows from the fixed leg are identical to those from a bond with coupon rate equal to the swap rate, while the cash flows from the floating leg are identical to those from a floating rate note. Thus, a swap can be viewed as exchanging a fixed rate coupon bond for a floating rate note.[5]

[3] A number of recent papers consider the effects of counterparty default risk in the valuation of interest-rate swaps. For example, see Solnik (1990), Sundaresan (1991), Cooper and Mello (1991), Rendleman (1992), Abken (1993), Sorensen and Bollier (1994), Hull and White (1995), Jarrow and Turnbull (1995), Li (1995), Duffie and Huang (1996), Jarrow, Lando, and Turnbull (1997), and Huge and Lando (1998).

[4] There is an extensive literature addressing the effects of asymmetric information and adverse selection on financial markets. Important examples include Glosten and Milgrom (1985), Kyle (1985), Admati and Pfleiderer (1988), and Brennan and Subrahmanyam (1996).

[5] For discussions about the economic role that interest-rate swaps play in financial markets, see Bicksler and Chen (1986), Turnbull (1987), Smith, Smithson, and

At the time a swap is initiated, the coupon rate on the fixed leg of the swap is specified. Intuitively, this rate is chosen to make the present value of the fixed leg equal to the present value of the floating leg. To illustrate how the fixed rate is determined, designate the current date as time zero, the final maturity date of the swap as time T, and let $D(t, N)$ denote the value at time t of a discount bond with arbitrary maturity N. The fixed rate at which a new swap with maturity T can be executed is known as the constant maturity swap rate and we denote it by $F(0, 0, T)$, where the first argument refers to time zero, the second argument denotes the start date of the swap which is time zero for a standard swap, and T is the final maturity date of the swap. Once a swap is executed, then fixed payments of $F(0, 0, T)/2$ are made semiannually at times $.50, 1.00, 1.50, \ldots, T - .50$, and T. Floating payments are made quarterly at times $.25, .50, .75, \ldots, T - .25$, and T and are equal to $d/360$ times the three-month Libor rate at the beginning of the quarter, where d is the number of days during the quarter. This feature is termed setting in advance and paying in arrears. Abstracting from credit issues, a floating rate note paying three-month Libor quarterly must be worth par at each quarterly Libor reset date. Since the initial value of a swap is zero, the initial value of the fixed leg must also be worth par. Setting the time-zero values of the two legs equal to each other and solving for the swap rate gives

$$F(0, 0, T) = 2\left[\frac{1 - D(0, T)}{A(0, 0, T)}\right], \tag{1}$$

where $A(0, 0, T) = \sum_{i=1}^{2T} D(0, i/2)$ is the present value of an annuity with first payment six months after the start date and final payment at time T. Swap rates are continuously available from a wide variety of sources for standard swap maturities such as 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, and 30 years.

For many swaptions, the underlying swap has a forward start date. In a forward swap with a start date of τ, fixed payments are made at time $\tau + .50, \tau + 1.00, \tau + 1.50, \ldots, T - .50$, and T and floating rate payments are made at times $\tau + .25, \tau + .50, \tau + .75, \ldots, T - .25$, and T. At the start date τ, the value of the floating leg equals par. Discounting this time-τ value back to time zero implies that the time-zero value of the floating cash flows is $D(0, \tau)$. Since the forward swap has a time-zero value of zero, the time-zero value of the fixed leg must also equal $D(0, \tau)$. This implies that the forward swap rate $F(0, \tau, T)$ must satisfy

$$F(0, \tau, T) = 2\left[\frac{D(0, \tau) - D(0, T)}{A(0, \tau, T)}\right]. \tag{2}$$

Wakeman (1988), Wall and Pringle (1989), Macfarlane, Ross, and Showers (1991), Sundaresan (1991), Litzenberger (1992), Sun, Sundaresan, and Wang (1993), and Gupta and Subrahmanyam (2000).

After a swap is executed, the coupon rate on the fixed leg may no longer equal the current market swap rate and the value of the swap can deviate from zero. Let $V(t, \tau, T, c)$ be the value at time t to the counterparty receiving fixed in a swap with forward start date $\tau \geq t$ and final maturity date T, where the coupon rate on the fixed leg is c. The value of this forward swap is given by

$$V(t, \tau, T, c) = \frac{c}{2} \sum_{i=1}^{2(T-\tau)} D(t, \tau + i/2) + D(t, T) - D(t, \tau), \tag{3}$$

where the first two terms in this expression represent the value of the fixed leg of the swap, and the third term is the present value of the floating leg which will be worth par at time τ. For $t > \tau$, the swap no longer has a forward start date and the value of the swap on semiannual fixed coupon payment dates is given by the expression

$$V(t, \tau, T, c) = \frac{c}{2} \sum_{i=1}^{2(T-t)} D(t, \tau + i/2) + D(t, T) - 1. \tag{4}$$

Note that in either case, the value of the swap is just a linear combination of zero-coupon bond prices.

2.3 European Swaptions.

There are two basic types of European swaptions. The first is the option to enter a swap and receive fixed. For example, let τ be the expiration date of the option, c be the coupon rate on the swap, and T be the final maturity date on the swap. The holder of this option has the right at time τ to enter into a swap with a remaining term of $T - \tau$ and receive the fixed annuity of c. Since the value of the floating leg will be par at time τ, this option is equivalent to a call option on a bond with a coupon rate of c and a remaining maturity of $T - \tau$ where the strike price of the call is \$1. Alternatively, this can be viewed as a derivative with payoff at time τ equal to $\max(0, V(\tau, \tau, T, c))$ This option is generally called a τ into $T - \tau$ receivers swaption, where τ is the maturity of the option and $T - \tau$ is the tenor of the underlying swap. This swaption is also known as a τ by T receivers swaption. Note that if the option holder is paying fixed at rate c in a swap with a final maturity date of T, then exercising this option has the effect of canceling the original swap at time t since the two fixed and two floating legs cancel each other out. Observe, however, that when the option is used to cancel the swap at time τ, the current fixed for floating coupon exchange is made first.

The second type of swaption is the option to enter a swap and pay fixed, and the cash flows associated with this option parallel those described above. An option which gives the option holder the right to enter into a swap at time τ with final maturity date at time T and pay fixed is generally termed a τ into $T - \tau$ or a τ by T payers swaption. Again, this option is equivalent to a put option on a coupon bond where the strike price is the value of the floating leg at time τ of \$1. Alternatively, the payoff at time τ can be expressed as $\max\big(0, -V(\tau, \tau, T, c)\big)$. A τ into $T-\tau$ payers swaption can be used to cancel an existing swap with final maturity date at time T where the option holder is receiving fixed at rate c.

Interest rate caplets can also be viewed as special cases of European swaptions. Many financial market participants enter into financial contracts where they pay or receive cash flows that are tied to some floating rate such as Libor. To hedge the risk created by the variability of the floating rate, firms often use interest rate caps or floors. A cap gives its holder a series of European call options or caplets on the Libor rate, where each caplet has the same strike price as the others, but a different expiration date. Similarly, floors represent a portfolio of individual floorlets or put options on the Libor rate. Note that the six-month Libor rate can be expressed as $F(t, \tau, \tau + /2)$. Because of this, it is easily shown that the payoff on a caplet on the six-month Libor rate can be expressed as $\max(0, -V(\tau, \tau, \tau + 1/2, c))$.[6] Thus, caps on six-month Libor can be viewed as simple portfolios of European options on six-month swaps.[7]

From the symmetry of the European payoff functions, it is easily shown that a long position in a τ into $T - \tau$ receivers swaption and a short position in a τ into $T - \tau$ payers swaption with the same coupon have the same payoff as receiving fixed in a forward swap with start date τ and coupon rate c. Specifically,

$$V(\tau, \tau, T, c) = \max\big(0, V(\tau, \tau, T, c)\big) - \max\big(0, -V(\tau, \tau, T, c)\big). \tag{5}$$

A standard no-arbitrage argument gives the receivers/payers parity result that at time t, $0 \leq t \leq \tau$, the value of the forward swap must equal the value of the receivers swaption minus the value of the payers swaption. When the coupon rate c is also equal to the forward swap rate $F(0, \tau, T)$, the forward swap is worth zero and the receivers and payers swaptions have identical values. In this case, the swaptions are said to be at the money forward.

2.4 American Swaptions.

Although there are a number of different variations, the most common type of American-style swaption is the T noncall τ structure. A T noncall τ receivers swaption gives the option holder the right to enter into a swap and receive fixed at any

[6]Libor is quoted on an actual/360 basis. For notational convenience, we do not adjust the swaption valuation formulas in the text for this daycount convention. In the computations, however, we explicitly adjust for the Libor daycount convention. For a discussion of caps and floors, see Longstaff, Santa-Clara, and Schwartz (2000).

[7]For many currencies, the market convention is for the cap to be on the three-month Libor rate. In some markets, however, caps may be on the six-month Libor rate. For example, Yen caps with maturities greater than one year are usually on the six-month Libor rate.

of the fixed coupon payment dates $\tau, \tau + .50, \tau + 1.00, \ldots, T - 1.00$, and $T - .50$. Similarly, a T noncall τ payers swaption gives the option holder the right to enter a swap and pay fixed at the same coupon payment dates. As before, either of these structures can be used to cancel existing swaps at any of these coupon payment dates after making the coupon exchange for that payment date. Since the underlying swap terminates at time T, the value of an American swaption converges to zero at time T. This implies that an American swaption should be exercised at time $T - .50$ if it is in the money. Hence, the value of an American swaption is identical to that of an equivalent European swaption at any time after the second to last coupon payment date for the underlying swap. These options are sometimes known as Bermuda swaptions, deferred American swaptions, or discrete American swaptions; for simplicity, we refer to them as American-style swaptions.

As with traditional American options, the value of an American-style swaption is greater than or equal to the value of its European counterpart. Furthermore, consider a standard T noncall τ structure. Exercise dates for this swaption range from time τ to time $T - .50$. For each of these exercise dates, we can find a European swaption with the same exercise date on the same underlying swap. Designate the set of these European swaptions as the corresponding European swaptions. Standard no-arbitrage results can be used to show that the value of the American-style swaption must be greater than or equal to the maximum of the values of all corresponding European swaptions.

2.5 Market Valuation Models.

As in most option markets, the convention in the European swaptions market is to quote prices in terms of their implied volatility relative to a standard pricing model. In trading European swaptions, prices are quoted as implied volatilities relative to the Black (1976) model as applied to the forward swap rate. To illustrate this, consider a τ into $T - \tau$ European receivers swaption where the fixed coupon rate equals c. Under the assumption that the forward swap rate follows a lognormal process under the pricing measure, the Black model implies that the value of this swaption at time zero is

$$\frac{1}{2} A(0, \tau, T) \left[F(0, \tau, T)\ N(d)\ -\ c\ N(d - \sigma\sqrt{\tau}) \right], \tag{6}$$

where

$$d = \frac{\ln(F(0, \tau, T)/c) + \sigma^2 \tau/2}{\sigma\sqrt{\tau}},$$

and where $N(\cdot)$ is the cumulative standard normal distribution function and σ is the volatility of the logarithm of the forward swap rate.[8] The value of the payers

[8] Smith (1991) describes the application of the Black (1976) model to European

swaption is given by the parity result. In the special case where the European receivers swaption is at the money forward, $c = F(0, \tau, T)$ and equation (6) reduces to

$$\big(D(0,\tau) - D(0,T)\big) \left[N\left(\frac{\sigma\sqrt{\tau}}{2}\right) - N\left(-\frac{\sigma\sqrt{\tau}}{2}\right)\right]. \tag{7}$$

Since this receivers swaption is at the money forward, the value of the corresponding payers swaption is identical. When an at-the-money forward swaption is quoted at an implied volatility of σ, the actual price paid by the purchaser of the swaption is given by substituting σ into equation (7).

Although there is no market standard for pricing American swaptions, extensive conversations with numerous brokers and dealers indicate that most Wall Street firms use some form of a single-factor Black, Derman, and Toy (1990) model in valuing their swaption positions and making exercise decisions. For example, the Black-Derman-Toy model is the default valuation model for American-style swaptions in the widely-used Bloomberg system.[9] In the Black-Derman-Toy model, the short-term rate r follows the dynamic process

$$\frac{dr}{r} = (\mu(t) + s'(t)/s(t)\ \ln r)\ dt + s(t)\ dZ, \tag{8}$$

where $\mu(t)$ is a drift function, $s(t)$ is a volatility function, and Z is a standard Brownian motion. This model is calibrated by fitting the drift function $\mu(t)$ and the volatility function $s(t)$ to match both the initial term structure and some subset of European swaption volatilities. Often $s(t)$ is chosen to be constant, and the model is calibrated to match the price of the τ into $T - \tau$ European swaption corresponding to the first exercise date of the American-style swaption.[10]

3. A STRING MODEL OF THE TERM STRUCTURE

In a series of recent papers, Kennedy (1994, 1997), Santa-Clara and Sornette (2000), Goldstein (2000), Longstaff, Schwartz, and Santa-Clara (2000), and Longstaff and

swaptions. Jamshidian (1996) and Brace, Gatarek, and Musiela (1997) demonstrate that the Black (1976) model for swaptions can be derived within an internally-consistent no-arbitrage model of the term structure.

[9]Other models that are used in practice include Black and Karasinski (1991) and Hull and White (1990). Both of these are single-factor models of the short-term rate and are closely related to the Black-Derman-Toy model.

[10]The constant volatility version of the Black-Derman-Toy model is designated the lognormal model in the Bloomberg system.

Schwartz (2000) model the evolution of the term structure as a stochastic string. In this approach, each point along the term structure is a distinct random variable with its own dynamics. Each point, however, is correlated with the other points along the term structure. Santa-Clara and Sornette show that this approach generalizes the Heath, Jarrow, and Morton (1992) framework while preserving its intuitive structure and appeal. By its nature, however, a string model has the advantage of being easier to calibrate to market data. Motivated by this literature, we develop a time-homogeneous string model of the term structure for the purpose of valuing interest rate options such as swaptions.

In this model, we take the six-month Libor forward rates out to 15 years, $F_i \equiv F(t, T_i, T_i + 1/2)$, $T_i = i/2$, $i = 1, 2, \ldots, 29$, to be the fundamental variables driving the term structure. Similarly to Black (1976), we assume that the risk-neutral dynamics for each forward rate are given by

$$dF_i = \alpha_i \ F_i \ dt + \sigma_i \ F_i \ dZ_i, \tag{9}$$

where α_i is an unspecified drift function, σ_i is a deterministic volatility function, dZ_i is a standard Brownian motion specific to this particular forward rate, and $t \leq T_i$.[11] Note that while each forward rate has its own dZ_i term, these dZ_i terms are correlated across forwards. The correlation of the Brownian motions together with the volatility functions determine the covariance matrix of forwards Σ. This is different from traditional implementations of multi-factor models which use several uncorrelated Brownian motions to shock each forward rate. This seemingly minor distinction actually has a number of important implications for the estimation of model parameters from market data.

To model the covariance structure among forwards in a parsimonious but economically sensible way, we make the assumption that the covariance between dF_i/F_i and dF_j/F_j is completely time homogeneous in the sense that it depends only on $T_i - t$ and $T_j - t$. Although the assumption of time homogeneity imposes additional structure on the model, it has the advantage of being more consistent with traditional dynamic term structure models in which interest rates are determined by the fundamental state of the economy. In addition, time homogeneity facilitates econometric estimation because of the stationarity of the model's specification. Furthermore, time homogeneity is intuitively appealing from the perspective of reflecting the fundamental economics of the problem. Because our ultimate objective is to apply the model to discretely-observed data, we make the simplifying assumption that these covariances are constant over six-month intervals. With these assumptions, the

[11]We assume that the initial value of F_i is positive and that the unspecified α_i terms are such that standard conditions guaranteeing the existence and uniqueness of a strong solution to equation (9) are satisfied. These conditions are described in Karatzas and Shreve (1988, Chapter 5). In addition, we assume that α_i is such that F_i is non-negative for all $t \leq T_i$.

problem of capturing the covariance structure among forwards reduces to specifying a 29 by 29 time-homogenous covariance matrix Σ.

One of the key differences between this string model and traditional multi-factor models is that our approach allows the parameters of the model to be uniquely identified from market data. For example, if there are N forward rates, the covariance matrix Σ has only $N(N+1)/2$ distinct parameters. Thus, market prices of fixed-income derivatives contain information on at most $N(N+1)/2$ covariances, and no more than $N(N+1)/2$ parameters can be uniquely identified from the market data. Since the string model is described by Σ, the parameters of the model are econometrically identified. In contrast, a typical implementation with constant coefficients of a traditional N-factor model of the form

$$dF_i = \alpha_i F_i dt + \sigma_{i1} F_i dZ_1 + \sigma_{i2} F_i dZ_2 + \ldots + \sigma_{iN} F_i dZ_N, \tag{10}$$

where the Brownian motions are uncorrelated, would require N parameters for each of the N forwards, resulting in a total of N^2 parameters. Given that there are only $N(N+1)/2 < N^2$ separate covariances among the forwards, the general specification in equation (10) cannot be identified using market information unless additional structure is placed on the model. Similar problems also occur when there are fewer factors than forwards. By specifying the covariance or correlation matrix among forwards directly, string models avoid these identification problems. String models also have the advantage of being more parsimonious. For example, up to $N \times K$ parameters would be needed to specify a traditional K-factor model. In contrast, only $K(K+1)/2$ parameters would be needed to specify a string model with rank K.

Although the string is specified in terms of the forward Libor rates, it is much more efficient to implement the model using discount bond prices. By definition,

$$F_i = 2\left[\frac{D(t, T_i)}{D(t, T_i + 1/2)} - 1\right]. \tag{11}$$

Thus, the forward rates F_i can all be expressed as functions of the vector of discount bond prices with maturities $.50, 1, \ldots, 15$. Conversely, these discount bond prices can be expressed as functions of the string of forward rates, assuming that standard invertibility conditions are satisfied.[12] Applying Itô's Lemma to the vector D of discount bond prices gives

$$dD = r\ D\ dt + J^{-1}\ \sigma\ F\ dZ, \tag{12}$$

[12] The primary condition is that the determinant of the Jacobian matrix for the mapping from discount bond prices to forward swap rates be non-zero. If this condition is satisfied, local invertibility is implied by the Inverse Function Theorem.

where r is the spot rate, $\sigma\ F\ dZ$ is the vector formed by stacking the individual terms $\sigma_i(t, T_i)\ F_i\ dZ_i$ in the forward rate dynamics in equation (9), and J^{-1} is the inverse of the Jacobian matrix for the mapping from discount bond prices to forward rates. Since each forward depends only on two discount bond prices, this Jacobian matrix has the following simple banded diagonal form.[13]

$$J = \begin{bmatrix} -\frac{2\,D(.50)}{D^2(1.00)} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{2}{D(1.50)} & -\frac{2\,D(1.00)}{D^2(1.50)} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{2}{D(2.00)} & -\frac{2\,D(1.50)}{D^2(2.00)} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{2}{D(14.50)} & -\frac{2\,D(14.00)}{D^2(14.50)} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{2}{D(15.00)} & -\frac{2\,D(14.50)}{D^2(15.00)} \end{bmatrix}$$

The drift term rD in equation (12) does not depend on the unspecified drift term α_i in equation (9). This is because discount bonds are traded assets in this complete markets setting and their expected return equals the spot rate under the risk-neutral measure. This approach allows us to avoid specifying the complicated drift term α_i, making the model much easier to parameterize and simulate than formulations based entirely on forward rates. Again, since our objective is a discrete-time implementation of this model, we make the simplifying assumption that r equals the yield on the shortest-maturity bond at each time period.[14]

The dynamics for D in equation (12) provide a complete specification of the evolution of the term structure. This string model is arbitrage free in the sense that it fits the initial term structure exactly and the expected rate of return on all discount

[13]For notational simplicity, discount bonds are expressed as functions of their maturity date in the Jacobian matrix. The Jacobian matrix represents the derivative of the 29 forwards $F_{.50}$, $F_{1.00}$, $F_{1.50}$, $\dots$, $F_{14.50}$ with respect to the discount bond prices $D(1.00)$, $D(1.50)$, $D(2.00)$, $\dots$, $D(15.00)$. Since $\sigma(T_i - t) = 0$ for $T_i \leq .50$, $D(.50)$ is not stochastic and does not affect the diffusion term in equation (12).

[14]This discretization assumption has little effect on the results. This approach gives values for European swaptions that are virtually identical to those implied by their closed-form solutions.

bonds equals the spot rate under the risk-neutral pricing measure. The model can also be easily extended to allow other interest rate dynamics than those given in equation (9). For example, we have also implemented the string model for the case where forward rates follow a constant elasticity of variance (CEV) process similar to Chan, Karolyi, Longstaff, and Sanders (1992). The results from this specification are virtually the same as those shown in this paper.[15] To complete the parameterization of the model, we need only specify Σ in a way that matches the market prices of interest rate options or the historical behavior of forward rates.

4. PARAMETERIZING THE MODEL

Rather than specifying the covariance matrix Σ exogenously, our approach is to solve for the implied matrix Σ that best fits the market prices of European swaptions and interest rate caps. The data used in estimating Σ consists of midmarket broker quotations for European swaptions and caps as reported by the Bloomberg system at the close of trading on July 2, 1999. These quotations are shown in Table 1.[16]

In solving for the implied covariance matrix, we note that a covariance matrix must be positive definite (or at least positive semidefinite) to be well defined. This means that care must be taken in designing the algorithm by which the covariance matrix is implied from the data to insure than this condition is satisfied. Standard results in linear algebra imply that a matrix is positive definite if, and only if, all eigenvalues of the matrix are positive.

Motivated by this necessary and sufficient condition, we use the following procedure to specify the implied covariance matrix. First, we estimate the historical correlation matrix of percentage changes in forward rates H from a time series of forward rates. Specifically, we obtain month-end Libor and swap rates from Bloomberg for the period from January 1989 to June 1999. Using a cubic spline, we estimate discount bond prices for each date and compute forward rates. We then estimate

[15] A more general approach would be to incorporate stochastic volatility into the string model framework by allowing the volatility parameters σ_i to vary over time according to some stochastic process. This would be consistent with the growing body of empirical evidence documenting that interest-rate volatility is stochastic. For example, see Brenner, Harjes, and Kroner (1996), Anderson and Lund (1997), Koedijk, Nissen, Schotman, and Wolff (1997), and Ball and Torous (1999). In addition, time series of both implied cap and swaption volatilities display persistent variation in their values; see Longstaff, Santa-Clara, and Schwartz (2000). Although the extension to stochastic volatility is beyond the scope of this paper, we note that the string model framework can easily accommodate stochastic volatility by either appending the dynamics for individual σ_i terms or by introducing additional factors driving common variation in the σ_i terms.

[16] We also replicate our results using data from a number of other dates during 1997, 1998, and 1999. The results are virtually the same as those reported.

the sample correlation matrix from these forward rates.[17]

We then decompose the historical correlation matrix into its spectral representation $H = U\Lambda U'$, where U is the matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues. Finally, we make the identifying assumption that the implied covariance matrix is of the form $\Sigma = U\Psi U'$, where Ψ is a diagonal matrix with non-negative elements. This assumption places an intuitive structure on the space of admissible implied covariance matrices.[18] Specifically, if the eigenvectors are viewed as factors, then this assumption is equivalent to assuming that the factors that generate the historical correlation matrix also generate the implied covariance matrix, but that the implied variances of these factors may differ from their historical values. Viewed this way, the identification assumption is simply the economically intuitive requirement that the market price interest rate options based on the factors which drive term structure movements.

Given this specification, the problem of finding the implied covariance matrix reduces to solving for the implied eigenvalues along the main diagonal of Ψ that best fit the market data. Rather than estimating a full diagonal of implied eigenvalues, however, we estimate only four and set the remaining eigenvalues equal to zero. Thus, we restrict the implied covariance matrix to a four-factor model. This is motivated by the recent evidence of Longstaff, Santa-Clara, and Schwartz (2000) documenting that the implied covariance matrix for European swaptions is of rank four. This is also consistent with the extensive empirical literature on the factor structure of term structure movements that generally finds evidence of three to four statistically significant factors.[19]

We solve for the implied eigenvalues by standard numerical optimization where the objective function is the root mean squared error (RMSE) of the percentage differences between the market price and the string model price based on 10,000 simulations. In simulating correlated Brownian motions, we use antithetic variates to reduce simulation noise. The time homogeneity of the model is implemented in the following way. During the first six-month simulation interval, the full 29 by 29 versions of the matrices Σ and J are used to simulate the dynamics of the 29 forward

[17] We implement this procedure using the historical correlation matrix rather than the covariance matrix to simplify the scaling of implied eigenvalues. We have also implemented this procedure using the historical covariance matrix. Not surprisingly, the eigenvectors from the historical covariance matrix are very similar to those obtained from the historical correlation matrix.

[18] This assumption is equivalent to requiring that the historical correlation matrix H and the implied covariance matrix Σ commute, that is, $H\Sigma = \Sigma H$. We are grateful to Bing Han for this observation.

[19] There are also other reasons for considering lower-dimensional specifications of the covariance matrix. For example, lower-dimensional models have the advantage of being more econometrically efficient and there may be less risk of overfitting with a more parsimonious specification; see Dai and Singleton (1999).

rates. After six months, however, the first forward becomes the spot rate, leaving only 28 forward rates to simulate during the second six month period. Because of the time homogeneity of the model, the relevant 28 by 28 covariance matrix is given by taking the first 28 rows and columns of Σ; the last row and column are dropped from the covariance matrix Σ. Similarly, the first row and column are dropped from the Jacobian since they involve derivatives with respect to the first forward which has now become the spot rate. This process is repeated until the last six-month period when only the final forward rate remains to be simulated.

In this study, we use three different parameterizations of Σ to ensure that our results are robust. A European swaption can be viewed as an option on a portfolio of forward rates since forward swap rates can be closely approximated as linear combinations of individual forward rates. Because of this, market prices of swaptions contain significant amounts of information about the correlation matrix of forwards. In these three parameterizations, we make use of this information about the correlation matrix by fitting the model to a wide variety of different swaption and cap prices. Recall that an interest rate caplet can be viewed as a European swaption on a short-term swap.

In the first parameterization, which we designate the swaption-cap parameterization, we solve for the implied Σ that best fits the six caps and the 42 European swaptions with $T \leq 15$ shown in Table 1, where equal weight is given to caps and swaptions in the objective function. The swaption-cap model fits the market quite well and the RMSE taken over all 48 prices is only 3.866 percent. This RMSE is significantly smaller than the typical bid/ask spread for the caps and swaptions which is on the order to 6 to 8 percent of their value.

In the second parameterization, which we designate the swaption parameterization, we solve for the implied Σ that best fits the 42 European swaptions only. This approach is similar to that used in Longstaff, Santa-Clara, and Schwartz (2000). The swaption model fits the data very well and has a RMSE of only 2.336 percent, where the RMSE is taken over the 42 swaptions. One possible reason for why the model fits better when only swaptions are included is suggested by the results of Longstaff, Santa-Clara, and Schwartz who find evidence that pricing in the caps market is not always consistent with pricing in the swaptions market.

In the third parameterization, which we designate the swaption-caplet parameterization, we first use a cubic spline procedure to interpolate the cap volatilities in Table 1 and obtain estimates of individual caplet volatilities out to 15 years.[20] We then constrain the main diagonal of Σ to match these caplet volatilities, resulting in an exact fit to the market prices of the original caps. This is done by first recognizing that the squared volatility of the Nth caplet is the average value of the first N terms

[20]Since the longest maturity cap is 10 years, we estimate the caplet volatilities from 10 to 15 years by a simple linear extrapolation. Alternative approaches yield similar results to those reported in the paper.

along the diagonal, and then solving for each element along the diagonal sequentially by a simple bootstrap procedure. With this constraint on the main diagonal of the covariance matrix, the problem then reduces to solving for the four eigenvalues of the implied correlation matrix that best fit the 42 European swaptions. Thus, this approach results in the cap prices being fitted exactly while the swaption prices are fitted to minimize their RMSE. This parameterization results in a RMSE of 6.979 for the 42 swaptions. This is significantly higher than for the other two parameterizations, although still on the same order of magnitude as the bid/ask spreads for the swaptions.

5. OPTIMAL AND SUBOPTIMAL EXERCISE STRATEGIES AND THE VALUATION OF AMERICAN SWAPTIONS

In valuing American-style swaptions, we use the least squares Monte Carlo (LSM) simulation technique of Longstaff and Schwartz (2000). There are three basic motivations for this valuation methodology. First, standard binomial or finite difference techniques are not computationally feasible because of the high dimensionality of the string model. Second, the LSM algorithm has the important advantage of allowing us to distinguish between the dimensionality of the exercise policy used for American-style swaptions and the dimensionality of the factor structure of the term structure. This differentiation is generally not possible using standard numerical techniques for American options. Finally, Longstaff and Schwartz demonstrate that the LSM algorithm is accurate and computationally efficient.

The key to the LSM approach is the fact that at any exercise date, the optimal stopping strategy for an American option is determined by comparing the value of immediate exercise with the value of continuing to keep the option alive. From standard option pricing theory, however, this continuation value can be expressed as a conditional expectation under the risk-neutral measure. The LSM approach estimates this conditional expectation using the cross-sectional information about the term structure in the simulation. Specifically, the LSM approach regresses the discounted ex post cash flows from continuing along each path onto functions of the current values of the state variables. The fitted value from this regression is an efficient estimator for the conditional expectation function. Note that the estimated conditional expectation will typically be nonlinear in the state variables. Exercising the option whenever the immediate exercise value is greater than the estimated value of continuation defines a simple stopping time rule, which Longstaff and Schwartz (2000) show closely approximates the optimal stopping rule.[21]

[21]Longstaff and Schwartz present a number of numerical examples showing that the LSM algorithm results in American option values that are numerically indistinguishable from those implied by finite difference techniques. On the other hand, since Longstaff and Schwartz only offer limited convergence results in their paper, it is

Since the optimal exercise strategy at any exercise date is determined by the conditional expectation function, we can contrast single-factor and multi-factor models in terms of their implications for the conditional expectation function. One well-known property of single-factor term structure models is that they imply that changes in the term structure are perfectly correlated across maturities. It is important to observe that this feature is due entirely to the fact that the time series properties of the term structure are driven by a single Brownian motion, and has nothing to do with how the single-factor model is parameterized to match the cross section of current market prices. A direct implication of this perfect correlation is that once we condition on the current value of the underlying swap, all other information about the term structure becomes redundant in forming the conditional expectation.[22] Thus, the current value of the swap becomes a sufficient statistic for making exercise decisions in a single-factor model.[23]

If the term structure is actually driven by multiple factors, then other information besides the current value of the underlying swap becomes useful in forming the conditional expectation. In a multi-factor setting, the conditional expectation depends not only on the current value of the underlying swap, but on all forward values of the swap as well. Thus, because of the perfect correlation inherent in single-factor models, single-factor exercise decisions are made conditional on only a subset of the relevant term structure information. This is the sense in which single-factor models are myopic. In contrast, the optimal multi-factor strategy is determined by the conditional expectation function which is based on all of the term structure information.

Intuitively, this result parallels Merton (1973) who shows that in a single-factor setting, the optimal early exercise decision for an American option is determined entirely by whether the value of the underlying asset exceeds a critical threshold.

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important to recognize that the accuracy of the LSM algorithm for high-dimensional problems where finite difference techniques cannot be used as a benchmark has not yet been fully verified.

[22]Note that we could also use any other point along the curve as the conditioning variable since all points are perfectly correlated. Doing this, however, results in virtually identical results. Since we need to condition on the underlying value of the swap anyway to determine the immediate exercise value of the swaption, the value of the underlying swap is the logical choice for the conditioning variable.

[23]Using the LSM framework, this result is easily verified by simulating paths of the term structure using different single-factor models such as Vasicek (1977), Cox, Ingersoll, and Ross (1985), Black Derman and Toy (1990), or a single-factor Heath, Jarrow, and Morton (1992) model. In each single-factor model, we estimate the continuation value for an American-style swaption by regressing discounted ex post cash flows on a basis set of functions of the value of the underlying swap. We then reestimate the regression after adding forward values of the underlying asset as additional explanatory variables. A standard F test shows that the additional variables do not have incremental explanatory power in the regression.

Thus, by its nature, a single-factor model implies that the exercise boundary is one-dimensional. In contrast, when the term structure is driven by multiple factors, the optimal exercise boundary is actually a multi-dimensional surface. In general, the multi-dimensional optimal exercise boundary cannot be well approximated by the one-dimensional exercise boundary implied by a single-factor model.

An alternative way of seeing this is from the perspective of the hedging or replication strategy. Because of the perfect correlation inherent in single-factor models, single-factor models imply that an American-style swaption can be perfectly hedged by trading in the underlying swap. If there are multiple factors, however, then an American swaption cannot be completely hedged by any single-security hedging portfolio, no matter how elaborately the single-factor model is fitted to the cross section of market prices. This latter point is particularly important since some practitioners believe that since a cross section of market option prices can be fit arbitrarily well in a single-factor model by simply adding extra parameters, the American option values implied by the fitted single-factor model must be accurate and the implied hedging strategy sound. It is clear, however, that fitting a model to a cross section of European swaption prices does not guarantee that the model will capture the time series properties or dynamics of the term structure. Because of the intertemporal nature of the American swaption exercise problem, however, the optimal exercise decision depends crucially on the dynamics of the term structure. Thus, by failing to capture the dynamic behavior of the term structure, single-factor models inherently miss a key determinant of the optimal exercise strategy.

To study the costs of using single-factor models to make exercise decisions for American-style swaptions when the term structure is actually driven by multiple factors, we use the following approach. First, we simulate paths of the term structure using the four-factor string model. We then solve for the single-factor stopping rule by including only functions of the current value of the underlying swap as explanatory variables in the LSM regressions.[24] Note that in doing this, we are giving the single-factor model its best chance since we allow the conditional expectation function to be estimated with knowledge of the actual structure of the economy over time by using the actual paths of the term structure generated by the multi-factor model. This is equivalent to allowing the single-factor model to be recalibrated at each exercise date to the current market values of all European interest rate derivatives. This clearly parallels the way an actual swaptionholder would use a single-factor model over time; the swaptionholder would need to correct for the model's failure to capture interest dynamics by recalibrating it at each exercise date before evaluating whether exercise was optimal. In this sense, the single-factor stopping rule we obtain is the best of all possible single-factor exercise policies since it is based on the actual term structure dynamics. We then apply the single-factor exercise strategy to paths

[24]We follow Longstaff and Schwartz (2000) in using the first three powers of the value of the underlying swap as basis functions. We also use a variety of other types and numbers of basis functions. The results are virtually identical to those reported.

generated by the multi-factor model and solve for the present value of the cash flows generated. Finally, we compare this present value with the present value obtained from following the optimal multi-factor strategy.

It is important to stress that the correct way to study the costs of following single-factor strategies in a multi-factor world is by comparing the present values of the cash flows obtained by following the stopping rules on a common set of paths of the term structure. A number of recent practitioner papers addressing this issue fail to do this, and come to the incorrect conclusion that American-style swaptions can actually be more valuable using a single-factor model than using a multi-factor model. These papers, however, suffer from several shortcomings. First, these papers do not hold fixed the dynamics of the term structure in their analysis and then compare results across different calibrations. This is the well-known problem of comparing across different equilibria. This comparison could only be relevant if the factor structure of the economy were to shift from a multi-factor to a single-factor structure. Second, these papers confuse the swaption value that comes out of the single-factor model with the present value that would actually be achieved from following the single-factor strategy in a multi-factor world. If the term structure is driven by multiple factors, then the American swaption value implied by the single-factor model has little relation to the present value of the actual cash flows that will be realized over time from following the single-factor exercise strategy. This is just a simple corollary of the well-known fact that if the dynamics of a model are misspecified, then the actual present value achieved by following the hedging strategy implied by the model need not equal the value given by the model. For many applications where the goal is to value European options, this bias can go either way and the realized present value may be higher or lower than the model price. Our key point, however, is that the American problem is much less forgiving; if the stopping rule is based on a misspecified single-factor model, the stopping rule will be suboptimal and the present value of realized cash flows will be less than if the optimal multi-factor model is followed.[25]

To illustrate the point that if a model is misspecified, then the value it gives for American options need not equal the present value of cash flows obtained by following the implied exercise strategy, consider the following example. Let X and Y denote the prices of two risky assets. One-period and two-period European exchange options are available in the market with payoffs of $\max(0, X_1 - Y_1)$ and $\max(0, X_2 - Y_2)$ respectively. Assuming that $r = .10$, the two assets have a correlation of .98, each has a current price of 100, and that the volatility of both assets is .10, the standard Margrabe (1978) model implies prices for these options of .80 and 1.13 respectively. Now assume that we want to value an American exchange option that is exercisable at times $t = 1$ and $t = 2$. It is easily shown that early exercise of this American

[25]Dybvig (1988) addresses a similar issue in the context of dynamic optimal portfolio choice problems. He shows that some common portfolio strategies fail to be optimal because they are incompletely diversified over time.

exchange option is not optimal, and that the correct value is 1.13.

Now assume, however, than an investor has the mistaken belief that X is constant over time and that market prices of these options are driven by the single factor Y. This investor now views the one-period and two-period exchange options as simple put options on Y with strike prices of X_0, and the investor can match their market prices by assuming that the volatility of Y is .100 during the first year, and .152 during the second year. Given this calibration, the single-factor model implies that the value of the American exchange option is 1.37. When this investor arrives at $t = 1$, the value of X may have changed, and he will need to recalibrate the single-factor model to reflect the new price of the remaining European exchange option. It is easily shown that once this calibration is performed, the investor will never find it optimal to exercise early at time $t = 1$. Thus, the actual cash flows received by the investor will have present value of 1.13 since they are the same as under the optimal strategy. The bottom line is that although the misspecified single-factor model values the American exchange option at 1.37, the actual present value of the cash flows is only about 82 percent of the amount that the investor would have been willing to pay to acquire the American exchange option. Note that at both times $t = 0$ and $t = 1$, the single-factor model is calibrated to match the market prices of the European exchange options. Despite this, the misspecified single-factor model results in a valuation for the American exchange option that is severely upward biased.

The appendix presents a more extensive example of an American option on a coupon bond in a two-factor binomial term structure model. In this example, an investor using a single-factor model calibrated to the market prices of all European options values the American option at .514. Applying the single-factor model over time, however, results in cash flows that have a present value of .563. In this case, the value implied by the single-factor model is actually biased downward by about ten percent.

These two examples show clearly that if the term structure is driven by multiple factors, the American option values given by a misspecified single-factor model are meaningless since they do not equal the present value of the cash flows generated by following the single-factor, multi-factor, or any other strategy. Thus, comparing the value implied by a single-factor model with the value implied by a multi-factor model is a conceptually flawed exercise. Single-factor and multi-factor models can only be compared in terms of the present value of cash flows generated by their exercise strategies while holding fixed the term structure model.

The fact that American option values implied by a misspecified model are biased poses some subtle but important risks to option holders. For example, consider an investor who believes that the term structure is driven by a single-factor and calibrates his model to exactly match a cross section of European swaptions. Now imagine that the single-factor model just happens to match the current market price of an American swaption as implied by the true multi-factor model. This investor might well conclude that since his model matches both the European and American swaption prices, that his single-factor model is adequate. In fact, however, the cash flows generated by following the single-factor strategy would have a lower present value than the market price of the American swaption. The only clues that this investor might have that there was a problem with his model would be the frequent need to recalibrate, persistent hedging errors, and a general tendency for his portfolio to underperform expectations. This underperformance would appear inexplicable to the investor since the valuations implied by the single-factor would match the market. The problem, of course, is that the American swaption is only worth the market price to an investor who follows the optimal multi-factor strategy. Purchasing an American swaption at the market price is a negative NPV investment to an investor following a suboptimal single-factor strategy.

6. THE COST OF SUBOPTIMAL EXERCISE STRATEGIES

In this section, we study the costs of following single-factor exercise strategies for American-style swaptions when the term structure is driven by multiple factors. We report the results separately for each of the three parameterizations of the string model.

6.1 The Swaption-Cap Parameterization.

Using the swaption-cap parameterization of the string model, we simulate paths of the term structure and then apply the LSM model to value a variety of typical American-style swaption structures. To value the American-style swaptions using a one-factor model, we use only functions of the current value of the underlying swap as explanatory variables in the LSM regressions. As discussed, this reflects the way that a swaptionholder using a single-factor model and then recalibrating the model at each exercise date would make exercise decisions over time. To value the American-style swaption under the optimal four-factor strategy, we include the same functions of the underlying swap value as well as all forward values of the swap as explanatory variables in the LSM regressions. The cost to the swaptionholder of using the suboptimal single-factor strategy is then estimated by taking the difference between the values of the swaption given by following the four-factor and one-factor exercise strategies.

Define the American exercise premium to be the difference between the T noncall τ American swaption value and the value of the corresponding τ into $T-\tau$ European swaption. Table 2 reports the American exercise premia given by the one-factor and four-factor models, the difference between these premia, and the difference between these premia expressed as a percentage of the one-factor American exercise premium.

Several important results are shown in Table 2. First, the value of the swaption is always higher when the optimal strategy is followed than when the single-factor strategy is followed. This demonstrates clearly that even the best single-factor exercise strategy cannot approximate the optimal strategy accurately enough to avoid some erosion in the value of swaption cash flows to an optionholder. Second, the size of the difference between the American exercise premia can be large in economic terms. For example, the difference between the American exercise premia often exceeds ten cents per $100 notional and can even exceed twenty cents. Since a large percentage of American-style swaptions are created in conjunction with the issuance of corporate or agency debt, it is typical for these swaptions to have longer maturities. Even if the cost of following the single-factor strategy is only five to ten cents, however, a back of the envelope calculation using the notional amounts for swaptions reported by ISDA implies that the total present value economic cost to swaption holders of using single-factor models to exercise swaptions could be on the order of several billion dollars. Finally, the percentage differences between the American exercise premia can also be large. In some cases, these percentage differences exceed ten percent.

To provide some additional intuition about the costs of suboptimal exercise, it is also useful to compare the optimal and single-factor exercise strategies in terms of their implications for the risk-neutral probability of exercise, the timing of exercise, and the cash flows received by swaption holders at exercise. Table 3 provides summary statistics for these measures. In comparing the optimal and single-factor strategies, the paths of the term structure are held fixed. Thus, Table 3 reflects only the differences in the exercise strategies.

Table 3 shows that in every case, following the optimal strategy results in the swaption being exercised a higher percentage of the time. While the differences between these probabilities are only about one to two percent, these are easily shown to be highly statistically significant. Note that the probabilities of exercising these at-the-money-forward American-style swaptions are all greater than 50 percent. This follows because a swaption holder has multiple chances to exercise an American-style swaption. In contrast, the probability of exercising an at-the-money European swaption is only 50 percent.

Table 3 also shows that the optimal and single-factor strategies differ in their implications for the timing of exercise. Typically, the optimal strategy results in the swaption being exercised later on average. There are, however, exceptions to this rule. For example, the mean time to exercise for the 5 noncall 2 receivers swaption is slightly lower under the optimal strategy than under the one-factor strategy. The table also reports the percentage of times that the optimal strategy results in an exercise that is earlier than, at the same time as, or later than for the one-factor strategy. As shown, the optimal policy frequently results in the swaption being exercised later than for the single-factor strategy, although the reverse can be true. A careful analysis of the paths where the option is exercised earlier under the optimal policy than under the single-factor strategy reveals that these are paths where the forward values of the underlying swap were substantially different from the current value of the underlying swap. Thus, by myopically focusing only on the current value of the underlying swap, the single-factor strategy misses the possibility that the swaption is expected to move out of the money in the future and hence fails to accelerate the exercise of the swaption. This clearly shows the suboptimality of the myopic single-factor strategy.

The fact that the optimal exercise strategy implies that the swaption can be exercised either earlier or later than for the single-factor strategy indicates that there is no simple adjustment to the single-factor strategy which will make it approximate the optimal strategy. For example, a common rule of thumb on Wall Street is to wait until the underlying asset is slightly beyond the one-factor exercise boundary before exercising it. The reasoning here is that if there is a second factor, such as stochastic volatility, the option may be worth more than implied by a single-factor model. If so, then an optionholder should be less willing to lose the time value of an American option by exercising it early. This analysis indicates that this rule of thumb does not solve the problem; there is no way in which a single-factor strategy can be adjusted to overcome its suboptimality. The optimal multi-dimensional exercise boundary implied by the optimal strategy cannot be well approximated by the one-dimensional exercise boundary implied by a single-factor model.

Finally, Table 3 shows that the average (undiscounted) cash flow received at exercise is always higher under the optimal strategy than under the single-factor strategy. In some cases, the difference can be very substantial, and may be as large as 20 to 30 cents. Note, however, that there is no guarantee that the multi-factor strategy will outperform the single-factor strategy along each path. There are clearly a small percentage of paths where the single-factor strategy results in a higher ex post cash flow. This is to be expected, however, since the optimal strategy is only optimal ex ante. Taken altogether, these results indicate that the economic costs from following the single-factor model come from both a reduction in the percentage of times the swaption is exercised as well as from a decrease in the average cash flow realized from exercise.

6.2 The Swaption Parameterization.

Using the swaption parameterization of the string model, Table 4 reports the summary statistics for the one-factor and four-factor American exercise premia. The results are similar to those described above. As before, the four-factor American exercise premia are all greater than the one-factor American exercise premia. In addition, the economic magnitude of the differences in the premia is significant, although typically not quite as large as in the previous parameterization. In general, the values of the American exercise premia themselves are fairly consistent between this and the previous parameterization, providing evidence for the stability of the parameterization methodology.

Turning to the probabilities of exercise, timing of exercise, and size of the cash flows implied by the optimal and one-factor strategies, summary statistics are presented in Table 5. As before, the probability of exercise is always higher under the optimal strategy than under the one-factor strategy. The mean time to exercise is again typically greater under the optimal strategy than under the single-factor strategy. This time, however, the 10 noncall 2 and 15 noncall 1 receivers swaptions are exceptions. The optimal strategy can result in the swaption being exercised earlier than for the single-factor strategy as much as 10 percent of the time or more. Finally, Table 5 shows that the mean cash flow resulting from following the optimal strategy is always greater than that from following the single-factor strategy.

6.3 The Swaption-Caplet Parameterization.

By requiring that the main diagonal of the implied covariance matrix price the interpolated values of individual caplets exactly, the swaptions and caplets parameterization implicitly places significant weight on the caps market in the fitting procedure. Using the paths of the term structure generated by this parameterization, Table 6 reports summary statistics for the American exercise premia implied by the optimal and single-factor exercise strategies.

Although this parameterization is structurally quite different from the previous two parameterizations, the basic implications for the cost of following single-factor exercise strategies are very similar. The magnitude of the differences in the American exercise premia are large in economic terms and are typically greater than in the previous two parameterizations. Percentage differences are now often greater than ten percent.

Table 7 shows that the probability of exercising a swaption is again strictly greater when the optimal policy is followed and represents an important component of the cost of following suboptimal single-factor strategies. The mean times to exercise are now all greater under the optimal strategy than under the single-factor strategy, which contrasts with the results for the earlier parameterizations. However, the result that the optimal strategy can result in earlier exercise than would be the case for the one-factor strategy holds for this parameterization. The summary statistics for mean cash flows in Table 7 again show that the optimal strategy results in larger cash flows on average, although individual paths can result in smaller cash flows.

7. CONCLUSION

This paper studies the costs of following single-factor exercise strategies for American swaptions when the term structure is actually driven by multiple factors. A number of important contributions emerge from this study.

- We develop a simple yet fully-specified multi-factor string model of the term structure for valuing interest rate derivatives. The architecture of this time-homogeneous model is based on the covariance matrix of forward rates. This feature has the important advantage of allowing the model to be identified in an econometric sense since the number of parameters required to specify the model just equals the number of parameters that can be inferred from market data. This feature is not shared by many current forward-rate-based term structure models.

- We estimate the present value costs to the holder of an American swaption who uses a one-factor model to make exercise decisions when the term structure is driven by multiple factors. Even though the swaption holder is allowed to recalibrate the one-factor model at each exercise date, we find that the best one-factor strategy is very suboptimal. Based on current market statistics, the total present value costs of following suboptimal strategies implied by single-factor models could be on the order of several billion dollars.
- These results make clear that if the dynamic specification of a model does not match actual market dynamics, the American exercise strategy implied by the model will be suboptimal. This is true no matter how extensively the model is fitted to match a cross section of current option prices. Furthermore, if the dynamics of a model are misspecified, then American option values implied by the model will be biased estimates of the actual present value of cash flows generated by following the exercise strategy implied by the model.

These results make a strong case for moving beyond simplistic single-factor models to more realistic (and easier to calibrate) string models in fixed income markets. From a theoretical point of view, string models have many advantages as a valuation framework, and this paper demonstrates that string models can be easily implemented. From a practical perspective, string models have the potential advantage of providing swaptionholders a tractable and intuitive way of extracting more value from their American-style swaptions by following more optimal exercise strategies.

APPENDIX

As a second example of the need to specify a model that adequately captures the dynamics of the underlying asset in order to price American options, we look at the valuation of an American option on a coupon bond in a two-factor binomial term structure model. Figure 1 shows the tree followed by the short rate in this world. We let the short rate r be equal to the sum of two independent or uncorrelated state variables x and y. The process for x is given by the recursion $x(t+1) = x(t)\exp(0.2u)$, where u can take the values -1 or 1 with risk-neutral probability 0.5. The process for y is similarly given by the recursion $y(t+1) = y(t)\exp(0.3v)$, where v can take the values -1 or 1, independently of u, again with risk-neutral probability 0.5. The initial values of x and y are $x(0) = 0.02$ and $y(0) = 0.03$, corresponding to an initial short rate $r(0) = 0.05$. The tree shows the values of the zero coupon bonds with maturities at times 2 and 3, where the price at time t of the discount bond with maturity T is denoted by $D(t,T)$, for $T = 2,3$, as well as the ex-coupon price of a coupon bond with maturity at time 3 and coupon rate 0.05, denoted $C(t,3)$. The tree also shows the values of two European call options on the coupon bond, both with strike 1, notional 100, and maturities at times 1 and 2, denoted $E(t,T)$, for $T = 1,2$. The payoff to the first call is thus $100\max(C(1,3)-1,0)$ and the payoff of the second is $100\max(C(2,3)-1,0)$. Finally, the tree shows the price of an American call option on the coupon bond, with strike 1, notional 100, and maturity at time 2, denoted $A(t,2)$. This option can be exercised early at time 1. The value of the American option at time 0 is .563, and the option should be optimally exercised at time 1 in node D, or else only at maturity.

Consider now the case of an investor who uses a single-factor Black, Derman and Toy model to price this American option. In this model, the short rate is given by the recursion $r(t+1) = r(t)\exp(m(t)+s(t)u)$, where u can take the values -1 or 1 with probability 0.5. At time 0, the investor solves for the values of $m(0)$, $m(1)$, $s(0)$ and $s(1)$ that fit the observed prices of the bonds as well as the European options of Figure 1. The corresponding binomial tree is shown in Figure 2. Note that this model is not time homogeneous, as the drift and the volatility of the short rate change through time. This is of course needed to be able to fit the prices of two bonds and two European options. This investor would value the American option at .514, which is about 10 percent less than the true value of the option. At time 1, the investor needs to recalibrate the one-factor model. Figure 3 shows the one-period binomial trees that are fitted to each of the four possible states of the world. In each of the trees, the investor picks new values for $m(1)$ and $s(1)$ to fit the then observed values of the remaining bond and European option. These values of $m(1)$ and $s(1)$ are different from the values found at time 0 and reflect the poor performance of the one-factor model in capturing the dynamics of the term structure. This problem is often encountered in practice, when time-inhomogeneous, low-dimensional models require different parameter values each time they are recalibrated. This need to recalibrate through time will typically correspond to a mispricing of American options. In this example, the investor also decides to exercise the option at node D and hold it until maturity in the other nodes. Thus, by construction, the cash flows obtained by the myopic investor are the same as those obtained by the far-sighted investor, and so is the time 0 value of those cash flows. However, the myopic investor significantly misprices the American option at time 0. Note that the mispricing of the American option by the myopic investor can go in either direction. For different parameter values of the two-factor binomial model, we obtained cases where the one-factor value of the American option was greater than, less than, or equal to the present value of the cash flows generated from following the single-factor exercise strategy.

REFERENCES

Abken, P., 1993, Valuation of Default-Risky Interest-Rate Swaps, *Advances in Futures and Options Research* 6, 93-116.

Admati, A. and P. Pfleiderer, 1988, A Theory of Intraday Patterns: Volume and Price Variability, *Review of Financial Studies* 1, 3-40.

Andersen, L., and J. Andreasen, 2000, Factor Dependence of Bermudan Swaption Prices: Fact or Fiction?, Working Paper, General Re Financial Products.

Anderson, T. and J. Lund, 1997, Estimating Continuous Time Stochastic Volatility Models of the Short-Term Interest Rate, *Journal of Econometrics* 77, 343-377.

Balduzzi, P., S. Das, S. Foresi, and R. Sundaram, 1996 A Simple Approach to Three Factor Affine Term Structure Models, *Journal of Fixed Income* 6, 43-53.

Ball, C. and W. Torous, 1999, The Stochastic Volatility of Short-Term Interest Rates: Some International Evidence, *The Journal of Finance* 54, 2339-2359.

Bicksler, J. and A. H. Chen, 1986, An Economic Analysis of Interest Rate Swaps, *Journal of Finance* 41, 645-655.

Black, F., 1976, The Pricing of Commodity Contracts, *Journal of Financial Economics* 3, 167-179.

Black, F., E. Derman, and W. Toy, 1990, A One-Factor Model of Interest Rates and its Application to Treasury Bond Options, *Financial Analysts Journal*, (January-February), 33-39.

Black, F. and P. Karasinski, 1991, Bond and Option Pricing When Short Rates are Lognormal, *Financial Analysts Journal*, (July-August), 52-59.

Brace, A., and R. Womersley, 2000, Exact Fit to the Swaption Volatility Matrix Using Semidefinite Programming, Working Paper, University of New South Wales.

Brace, A., D. Gatarek, and M. Musiela, 1997, The Market Model of Interest Rate Dynamics, *Mathematical Finance* 7, 127-155.

Brennan, M. and A. Subrahmanyam, 1996, Market Microstructure and Asset Pricing: On the Compensation for Illiquidity in Stock Returns, *Journal of Financial Economics* 41, 441-464.

Brenner, R., R. Harjes, and K. Kroner, 1996, Another Look at Models of the Short-Term Interest Rate, *Journal of Financial and Quantitative Analysis* 31, 85-107.

Brown, S. and P. Dybvig, 1986, The Empirical Implications of the Cox, Ingersoll, Ross

Theory of the Term Structure of Interest Rates, *Journal of Finance* 41, 617-630.

Chan, K. C., A. Karolyi, F. Longstaff, and A. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance* 47, 1209-1227.

Chen, R.R., and L. Scott, 1993, Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates, *Journal of Fixed Income* 3, 14-31.

Cooper, I. and A. Mello, 1991, The Default Risk of Swaps, *Journal of Finance* 46, 597-620.

Cox, J., J. Ingersoll, and S. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385-407.

Dai, Q. and K. Singleton, 1999, Specification Analysis of Affine Term Structure Models, *The Journal of Finance*, forthcoming.

Duffie, D. and M Huang, 1996, Swap Rates and Credit Quality, *The Journal of Finance* 51, 921-949.

Dybvig, P., 1988, Inefficient Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market, *The Review of Financial Studies* 1, 67-88.

Gibbons, M. and K. Ramawswamy, 1993, A Test of the Cox, Ingersoll, and Ross Model of the Term Structure, *Review of Financial Studies* 6, 619-658.

Glosten, L. and P. Milgrom, 1985, Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders, *Journal of Financial Economics* 14, 71-100.

Goldstein, R., 2000, The Term Structure of Interest Rates as a Random Field, The Review of Financial Studies, forthcoming.

Gupta, A. and M. Subrahmanyam, 2000, An Empirical Investigation of the Convexity Bias in the Pricing of Interest Rate Swaps, *Journal of Financial Economics*, forthcoming.

Heath, D., R. Jarrow, and A. Morton, 1992, Bond Pricing and the Term Structure of Interest Rates, *Econometrica* 60, 77-106.

Huge, B., and D. Lando, 1998, Swap Pricing with Two-Sided Default Risk in a Rating-Based Model, Working Paper, University of Copenhagen.

Hull, J. and A. White, 1990, Pricing Interest Rate Derivative Securities, *The Review of Financial Studies* 3, 573-592.

Hull, J. and A. White, 1995, The Impact of Default Risk on the Prices of Options and Other Derivative Securities, *Journal of Banking and Finance* 19, 299-322.

Jamshidian, F., 1996, Libor and Swap Market Model and Measures II, Working Paper, Sakura Global Capital.

Jarrow, R., D. Lando, and S. Turnbull, 1997, A Markov Model for the Term Structure of Default Spreads, *Review of Financial Studies*, 481-523.

Jarrow, R. and S. Turnbull, 1995, Pricing Options on Financial Securities Subject to Credit Risk, *Journal of Finance* 50, 53-86.

Karatzas, I, 1988, On the Pricing of American Options, *Applied Mathematics and Optimization* 17, 37-60.

Karatzas, I. and S. Shreve, 1988, Brownian Motion and Stochastic Calculus, Springer Verlag Inc., New York, NY.

Kennedy, D., 1994, The Term Structure of Interest Rates as a Gaussian Random Field, *Mathematical Finance* 4, 247-258.

Kennedy, D., 1997, Characterizing Gaussian Models of the Term Structure of Interest Rates, *Mathematical Finance* 7, 107-118.

Knez, P., R. Litterman, and J. Scheinkman, 1994, Explorations into Factors Explaining Money Market Returns, *Journal of Finance* 49, 1861-1882.

Koedijk, K, F. Nissen, P. Schotman, and C. Wolff, 1997, The Dynamics of Short-Term Interest Rate Volatility Reconsidered, *European Finance Review* 1, 105-130.

Knez, P., R. Litterman, and J. Scheinkman, Explorations into Factors Explaining Money Market Returns, *The Journal of Finance* 49, 1861-1882.

Kyle, A., 1985, Continuous Auctions and Insider Trading, *Econometrica* 53, 1315-1335.

Li, H., 1995, Pricing of Swaps with Default Risk, Working Paper, Yale University.

Litterman, R. and J. Scheinkman, 1991, Common Factors Affecting Bond Returns, *The Journal of Fixed Income* 1, 54-61.

Litzenberger, R., 1992, Swaps: Plain and Fanciful, *The Journal of Finance* 42, 831-850.

Longstaff, F. and E. S. Schwartz, 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *The Journal of Finance* 47, 1259-1282.

Longstaff, F. and E. S. Schwartz, 2000, Pricing American Options by Simulation: A Simple Least Squares Approach, *The Review of Financial Studies*, forthcoming.

Longstaff, F., P. Santa-Clara, and E. S. Schwartz, 2000, The Relative Valuation of Caps and Swaptions: Theory and Empirical Evidence, Working Paper, UCLA.

Macfarlane, J., D. Ross, and J. Showers, 1991, The Interest Rate Swap Market: Yield Mathematics, Terminology, and Conventions, in *Interest Rate Swaps*, ed. Carl R. Beidleman, (Business One Irwin, Homewood, IL).

Marsh, T. and E. Rosenfeld, 1983, Stochastic Processes for Interest Rates and Equilibrium Bond Prices, *Journal of Finance* 38, 635-646.

Margrabe, W., 1978, The Value of an Option to Exchange One Asset for Another, *Journal of Finance* 33, 177-186.

Merton, R. C., 1973, The Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141-183.

Pearson, N. and T.-S. Sun, 1994, Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model, *Journal of Finance* 49, 1279-1304.

Piazzesi, M., 2000, An Econometric Model of the Yield Curve with Macroeconomic Jump Effects, Working Paper, Stanford University.

Rendleman, R., 1992, How Risks are Shared in Interest Rate Swaps, *The Journal of Financial Services Research*, 5-34.

Santa-Clara, P. and D. Sornette, 2000, The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks, *The Review of Financial Studies*, forthcoming.

Solnik, B, 1990, Swap Pricing and Default Risk: A Note, *Journal of International Financial Management and Accounting* 2, 79-91.

Sorensen, E. H., and T. F. Bollier, 1994, Pricing Swap Default Risk, *Financial Analysts Journal*, (May-June), 23-33.

Smith, D. R, 1991, A Simple Method for Pricing Interest Rate Swaptions, *Financial Analysts Journal*, (May-June), 72-76.

Smith, C. W., C. W. Smithson, and L. M. Wakeman, 1988, The Market for Interest Rate Swaps, *Financial Management* 17, 34-44.

Stambaugh, R., 1988, The Information in Forward Rates: Implications for Models of the Term Structure, *Journal of Financial Economics* 21, 41-70.

Sun, T., S. M. Sundaresan, C. Wang, 1993, Interest Rate Swaps: An Empirical Investigation, *Journal of Financial Economics* 36, 77-99.

Sundaresan, S. M., 1991, Valuation of Swaps, in Sarkis J. Khoury, ed., Recent Developments in International Banking and Finance, vol 5, S. J. Khoury, ed., New York:Elsevier Science Publishers.

Turnbull, S., 1987, Swaps: Zero-Sum Game?, *Financial Management* 16, 15-21.

Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177-188.

Wall, L. D. and J. J. Pringle, 1989, Alternative Explanations of Interest Rate Swaps: A Theoretical and Empirical Analysis, *Financial Management* 16, 15-21

Table 1

Broker Swaption and Cap Volatilities. This table shows mid-market implied volatilities for the indicated at-the-money-forward European swaptions and interest-rate caps for July 2, 1999 as reported by the Bloomberg system. Swaption maturity represents the number of years until expiration for the swaption. Swap tenor refers to the length in years of the swap that the swaptionholder enters into if the swaption is exercised. Cap maturity refers to the number of years until the maturity date of the final caplet.

	Swap Tenor						
Swaption Maturity	1	2	3	4	5	7	10
.5	16.8	16.8	16.6	16.4	16.3	16.2	16.2
1	16.9	16.7	16.5	16.2	16.0	15.8	15.7
2	17.0	16.7	16.5	16.2	15.8	15.6	15.4
3	16.9	16.5	16.3	16.0	15.6	15.2	15.0
4	16.8	16.4	16.1	15.7	15.4	14.9	14.6
5	16.6	16.1	15.7	15.3	15.0	14.4	14.0
7	15.2	14.4	14.0	13.7	13.3	12.9	12.5
10	13.1	12.4	11.9	11.5	11.2	10.8	10.3

	Cap Maturity					
	2	3	4	5	7	10
Cap Volatility	17.25	18.50	18.62	18.62	18.20	17.20

Table 2

Summary Statistics for American Exercise Premia from the Swaption-Cap Parameterization of the String Model. This table reports the values per $100 notional of the American exercise premia for the indicated T noncall τ American at-the-money-forward swaptions implied by the string model under the single-factor and four-factor exercise strategies. The American exercise premium is the difference between the American swaption value and the corresponding European swaption value. The percentage difference is computed relative to the one-factor American exercise premium. The model is parameterized by solving for the four implied eigenvalues that best fit the market prices of the 42 swaptions where the final maturity date of the underlying swap is less than or equal to 15 years and also to the market prices to two, three, four, five, seven, and ten year caps. The fitting procedure places 50 percent of the weight on the caps and 50 percent of the weight on the swaptions. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

T	τ	Swaption Type	One-Factor American Exercise Premium	Four-Factor American Exercise Premium	Difference	Percentage Difference
5	1	Recvr	.525	.542	.017	3.24
5	2	Recvr	.263	.272	.009	3.42
10	1	Recvr	1.799	1.846	.047	2.61
10	2	Recvr	1.222	1.279	.057	4.66
10	5	Recvr	.353	.381	.028	7.93
15	1	Recvr	3.116	3.236	.120	3.85
15	3	Recvr	1.758	1.851	.093	5.29
15	5	Recvr	1.018	1.086	.068	6.68
5	1	Payer	.681	.709	.028	4.11
5	2	Payer	.281	.301	.020	7.12
10	1	Payer	2.452	2.609	.157	6.40
10	2	Payer	1.590	1.715	.125	7.86
10	5	Payer	.436	.491	.055	12.61
15	2	Payer	4.232	4.460	.228	5.39
15	3	Payer	2.281	2.503	.222	9.73
15	5	Payer	1.283	1.424	.141	11.00

Table 3

Comparison of One-Factor and Four-Factor Exercise Strategies using Swaption-Cap Parameterization of String Model. This table reports summary statistics for the one-factor and four-factor exercise strategies. Total probability represents the total percentage of paths for which the swaption is exercised. The mean time to exercise is expressed in years. The earlier, same, and later statistics represent the percentage of times that following the four-factor exercise strategy results in an exercise earlier, at the same time, or later than when the one-factor strategy is followed. Mean cash flow at exercise is the average cash flow per $100 notional received when the swaption is exercised. The less, equal, and greater statistics represent the percentage of times that following the four-factor exercise strategy results in a cash flow that is less than, equal to, or greater than the cash flows that are received when the one-factor strategy is followed. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

			Probability of Exercise		Mean Time to Exercise		Four-Factor Strategy Results in Exercise			Mean Cash Flow at Exercise		Four-Factor Strategy Results in Cash Flow		
T	τ	Swaption Type	One-Factor	Four-Factor	One-Factor	Four-Factor	Earlier	Same	Later	One-Factor	Four-Factor	Less	Equal	Greater
5	1	Recvr	67.51	68.41	3.51	3.51	5.07	88.22	6.71	1.050	1.062	5.03	88.22	6.75
5	2	Recvr	66.06	66.82	3.93	3.92	4.08	90.65	5.27	.992	.998	4.14	90.65	5.21
10	1	Recvr	69.44	70.36	6.21	6.24	8.55	75.49	15.96	2.552	2.606	10.28	75.49	14.23
10	2	Recvr	69.42	70.13	6.48	6.50	7.09	78.31	14.60	2.636	2.693	8.90	78.31	12.79
10	5	Recvr	65.71	66.49	7.87	7.87	4.43	87.28	8.29	2.239	2.266	5.19	87.28	7.53
15	1	Recvr	71.57	72.38	8.96	9.02	10.01	67.18	22.81	3.997	4.154	12.97	67.18	19.85
15	3	Recvr	71.19	71.93	9.45	9.52	7.93	72.50	19.57	4.198	4.329	10.73	72.50	16.77
15	5	Recvr	69.28	70.17	10.35	10.39	6.85	77.24	15.91	4.068	4.162	8.86	77.24	13.90
5	1	Payer	64.36	65.49	3.74	3.74	8.08	80.83	11.09	1.149	1.168	8.60	80.83	10.57
5	2	Payer	59.88	60.87	4.13	4.13	5.31	87.14	7.55	1.019	1.033	5.48	87.14	7.38
10	1	Payer	68.22	69.65	6.62	6.62	12.28	64.94	22.78	3.014	3.152	14.57	64.94	20.49
10	2	Payer	66.41	67.77	6.89	6.89	10.53	68.28	21.19	2.923	3.043	12.94	68.28	18.78
10	5	Payer	59.92	61.02	8.23	8.23	6.42	81.82	11.76	2.509	2.561	7.51	81.82	10.67
15	1	Payer	71.92	73.06	9.13	9.13	14.09	56.41	29.50	4.803	5.076	17.89	56.41	25.70
15	3	Payer	69.64	70.79	9.75	9.75	10.63	62.08	27.29	4.678	4.961	14.95	62.08	22.97
15	5	Payer	66.59	67.78	10.68	10.68	8.62	68.64	22.74	4.556	4.754	12.33	68.64	19.03

Table 4

Summary Statistics for American Exercise Premia from the Swaption Parameterization of the String Model. This table reports the values per $100 notional of the American exercise premia for the indicated T noncall τ American at-the-money-forward swaptions implied by the string model under the single-factor and four-factor exercise strategies. The American exercise premium is the difference between the American swaption value and the corresponding European swaption value. The percentage difference is computed relative to the one-factor American exercise premium. The model is parameterized by solving for the four implied eigenvalues that best fit the market prices of the 42 swaptions where the final maturity date of the underlying swap is less than or equal to 15 years. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

T	τ	Swaption Type	One-Factor American Exercise Premium	Four-Factor American Exercise Premium	Difference	Percentage Difference
5	1	Recvr	.502	.506	.004	.80
5	2	Recvr	.241	.243	.003	1.24
10	1	Recvr	1.739	1.789	.051	2.93
10	2	Recvr	1.182	1.235	.053	4.48
10	5	Recvr	.331	.340	.009	2.72
15	1	Recvr	3.110	3.191	.081	2.60
15	3	Recvr	1.694	1.753	.059	3.48
15	5	Recvr	.993	1.039	.045	4.53
5	1	Payer	.658	.667	.010	1.52
5	2	Payer	.261	.270	.009	3.45
10	1	Payer	2.401	2.491	.091	3.79
10	2	Payer	1.527	1.612	.085	5.57
10	5	Payer	.406	.428	.023	5.67
15	2	Payer	4.217	4.376	.158	3.75
15	3	Payer	2.257	2.398	.141	6.25
15	5	Payer	1.242	1.354	.112	9.02

Table 5

Comparison of One-Factor and Four-Factor Exercise Strategies using Swaption Parameterization of String Model. This table reports summary statistics for the one-factor and four-factor exercise strategies. Total probability represents the total percentage of paths for which the swaption is exercised. The mean time to exercise is expressed in years. The earlier, same, and later statistics represent the percentage of times that following the four-factor exercise strategy results in an exercise earlier, at the same time, or later than when the one-factor strategy is followed. Mean cash flow at exercise is the average cash flow per $100 notional received when the swaption is exercised. The less, equal, and greater statistics represent the percentage of times that following the four-factor exercise strategy results in a cash flow that is less than, equal to, or greater than the cash flows that are received when the one-factor strategy is followed. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

			Probability of Exercise		Mean Time to Exercise		Four-Factor Strategy Results in Exercise			Mean Cash Flow at Exercise		Four-Factor Strategy Results in Cash Flow		
T	τ	Swaption Type	One-Factor	Four-Factor	One-Factor	Four-Factor	Earlier	Same	Later	One-Factor	Four-Factor	Less	Equal	Greater
5	1	Recvr	66.76	66.95	3.50	3.51	3.18	92.70	4.12	1.031	1.034	3.32	92.70	3.98
5	2	Recvr	65.05	65.19	3.93	3.94	2.07	94.69	3.24	.968	.970	2.44	94.69	2.87
10	1	Recvr	67.81	68.71	6.23	6.23	7.29	80.69	12.02	2.513	2.559	8.01	80.69	11.30
10	2	Recvr	67.92	68.55	6.49	6.48	6.14	82.99	10.87	2.608	2.652	7.00	82.99	10.01
10	5	Recvr	64.44	64.81	7.89	7.89	3.40	91.04	5.56	2.195	2.204	3.95	91.04	5.01
15	1	Recvr	70.37	71.23	9.00	8.98	10.56	70.19	19.25	3.998	4.092	12.87	70.19	16.94
15	3	Recvr	69.83	70.33	9.49	9.53	7.76	75.85	16.39	4.160	4.245	10.31	75.85	13.84
15	5	Recvr	68.01	68.49	10.41	10.44	6.01	80.81	13.18	4.043	4.106	8.04	80.81	11.15
5	1	Payer	63.94	64.26	3.74	3.75	4.84	88.37	6.79	1.131	1.138	5.36	88.37	6.27
5	2	Payer	59.69	59.94	4.12	4.12	3.08	92.06	4.86	.999	1.005	3.56	92.06	4.38
10	1	Payer	67.29	68.21	6.61	6.63	10.80	71.35	17.85	2.987	3.066	12.52	71.35	16.13
10	2	Payer	65.28	66.37	6.89	6.91	8.59	75.12	16.29	2.889	2.969	10.32	75.12	14.56
10	5	Payer	58.72	59.50	8.25	8.24	5.34	86.53	8.13	2.471	2.492	5.78	86.53	7.69
15	1	Payer	71.64	72.29	9.09	9.18	13.13	60.51	26.36	4.802	4.995	16.80	60.51	22.69
15	3	Payer	69.15	69.75	9.75	9.87	9.89	66.04	24.07	4.690	4.880	14.14	66.04	19.82
15	5	Payer	65.99	66.70	10.69	10.79	7.65	72.67	19.68	4.544	4.693	10.90	72.67	16.43

Table 6

Summary Statistics for American Exercise Premia from the Swaption-Caplet Parameterization of the String Model. This table reports the values per $100 notional of the American exercise premia for the indicated T noncall τ American at-the-money-forward swaptions implied by the string model under the single-factor and four-factor exercise strategies. The American exercise premium is the difference between the American swaption value and the corresponding European swaption value. The percentage difference is computed relative to the one-factor American exercise premium. The model is parameterized by solving for the four implied eigenvalues that best fit the market prices of the 42 swaptions where the final maturity date of the underlying swap is less than or equal to 15 years and also to the implied prices of 29 caplets with maturities ranging from .50 to 15.00 years. By construction, the fitting procedure fits the caplet prices exactly. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

T	τ	Swaption Type	One-Factor American Exercise Premium	Four-Factor American Exercise Premium	Difference	Percentage Difference
5	1	Recvr	.564	.611	.047	8.33
5	2	Recvr	.280	.309	.029	10.36
10	1	Recvr	1.809	1.881	.072	3.98
10	2	Recvr	1.218	1.307	.089	7.31
10	5	Recvr	.389	.449	.060	15.42
15	1	Recvr	2.965	3.060	.095	3.20
15	3	Recvr	1.696	1.820	.124	7.31
15	5	Recvr	1.032	1.136	.104	10.08
5	1	Payer	.721	.804	.083	11.51
5	2	Payer	.300	.351	.051	17.00
10	1	Payer	2.528	2.714	.186	7.36
10	2	Payer	1.636	1.822	.186	11.37
10	5	Payer	.491	.605	.114	23.22
15	2	Payer	4.144	4.368	.224	5.41
15	3	Payer	2.293	2.525	.232	10.12
15	5	Payer	1.336	1.541	.205	15.34

Table 7

Comparison of One-Factor and Four-Factor Exercise Strategies using Swaption-Caplet Parameterization of String Model. This table reports summary statistics for the one-factor and four-factor exercise strategies. Total probability represents the total percentage of paths for which the swaption is exercised. The mean time to exercise is expressed in years. The earlier, same, and later statistics represent the percentage of times that following the four-factor exercise strategy results in an exercise earlier, at the same time, or later than when the one-factor strategy is followed. Mean cash flow at exercise is the average cash flow per $100 notional received when the swaption is exercised. The less, equal, and greater statistics represent the percentage of times that following the four-factor exercise strategy results in a cash flow that is less than, equal to, or greater than the cash flows that are received when the one-factor strategy is followed. All values are calculated using the LSM algorithm based on 10,000 simulated paths of the term structure.

			Probability of Exercise		Mean Time to Exercise		Four-Factor Strategy Results in Exercise			Mean Cash Flow at Exercise		Four-Factor Strategy Results in Cash Flow		
T	τ	Swaption Type	One-Factor	Four-Factor	One-Factor	Four-Factor	Earlier	Same	Later	One-Factor	Four-Factor	Less	Equal	Greater
5	1	Recvr	68.60	69.64	3.51	3.55	5.62	79.65	14.73	1.081	1.117	7.95	79.65	12.40
5	2	Recvr	66.92	68.00	3.91	3.93	4.63	83.68	11.69	1.052	1.073	6.46	83.68	9.86
10	1	Recvr	71.49	72.13	6.12	6.22	7.25	72.35	20.40	2.541	2.617	10.81	72.35	16.84
10	2	Recvr	71.54	72.20	6.38	6.46	6.87	73.28	19.85	2.626	2.711	10.27	73.28	16.45
10	5	Recvr	67.95	69.00	7.82	7.84	5.29	80.19	14.52	2.221	2.281	7.40	80.19	12.41
15	1	Recvr	71.29	72.10	8.87	8.95	9.03	69.34	21.63	3.904	4.048	12.12	69.34	18.54
15	3	Recvr	71.18	72.14	9.37	9.45	7.43	71.98	20.59	4.086	4.243	10.41	71.98	17.61
15	5	Recvr	69.48	70.64	10.33	10.38	7.19	74.25	18.56	3.918	4.061	9.39	74.25	16.36
5	1	Payer	63.97	65.80	3.76	3.79	9.59	69.47	20.94	1.176	1.234	12.65	69.47	17.88
5	2	Payer	59.62	61.21	4.13	4.15	6.24	79.05	14.71	1.072	1.108	8.10	79.05	12.85
10	1	Payer	70.49	71.25	6.51	6.66	11.01	59.28	29.71	3.031	3.196	16.36	59.28	24.36
10	2	Payer	68.55	69.27	6.80	6.95	9.31	62.58	28.11	2.921	3.102	14.62	62.58	22.80
10	5	Payer	61.94	63.35	8.17	8.23	6.64	73.19	20.17	2.481	2.592	10.27	73.19	16.54
15	1	Payer	72.19	73.52	9.03	9.17	12.91	55.77	31.32	4.762	5.054	17.72	55.77	26.51
15	3	Payer	70.17	71.47	9.64	9.85	10.49	59.96	29.55	4.628	4.935	15.28	59.96	24.76
15	5	Payer	67.43	68.65	10.60	10.76	9.27	63.50	27.23	4.453	4.723	13.88	63.50	22.62

Figure 1

Two-Factor Binomial Tree for Pricing an American Option on a Coupon Bond. The tree below shows the paths of a two-factor binomial model of the short rate, r, under the risk-neutral probability measure. The short rate is equal to the sum of two state variables x and y, r(t) = x(t) + y(t). The process for x is given by the recursion x(t+1) = x(t) exp(0.2u), where u can take the values -1 or 1 with probability 0.5. The process for y is given by the recursion y(t+1) = y(t) exp(0.3v), where v can take the values -1 or 1, independently of u, with probability 0.5. The initial values of x and y are x(0) = 0.02 and y(0) = 0.03. The tree shows the values of the zero coupon bonds with maturities at times 2 and 3, where the price at time t of the discount bond with maturity T is denoted by D(t,T), for T = 2, 3, as well as the ex-coupon price of a coupon bond with maturity at time 3 and coupon rate 0.05, denoted C(t,3). The tree also shows the values of two European options on the coupon bond, with strike 1, a notional of 100, and maturities at times 1 and 2, denoted E(t,T), for T = 1, 2. Finally, the tree shows the price of an American option on the coupon bond, with strike 1, a notional of 100, with maturity at time 2, denoted A(t,2).

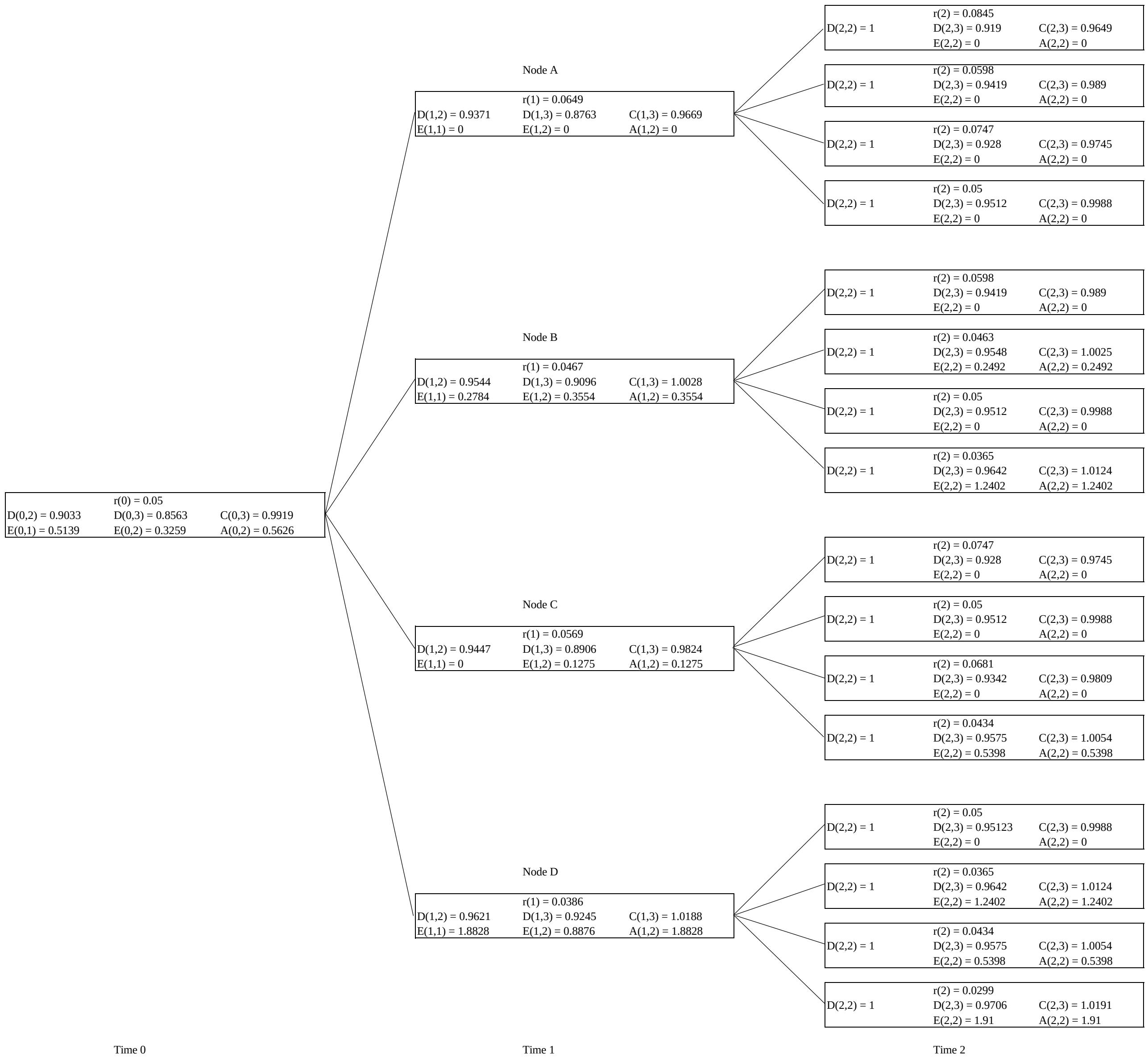

Figure 2

One-Factor Binomial Tree for Pricing an American Option on a Coupon Bond, Calibrated to Match the Bond Prices and European Option Prices at Time 0 of the Two-Factor Tree in Figure 1. The tree below shows the paths of a one-factor binomial model of the short rate, r, under the risk-neutral probability measure. The process for the short rate is given by the recursion r(t+1) = r(t) exp(m(t)+s(t)u), where u can take the values -1 or 1 with probability 0.5. The values of m and s are m(0) = 0.06124, m(1) = -0.081017, s(0) = 0.170083, and s(1) = 0.122065, which are chosen to fit the prices of bonds and European options observed at time 0 in the two-factor tree of figure 1. The tree shows the values of the zero coupon bonds with maturities at times 2 and 3, where the price at time t of the discount bond with maturity T is denoted by D(t,T), for T = 2, 3, as well as the ex-coupon price of a coupon bond with maturity at time 3 and coupon rate 0.05, denoted C(t,3). The tree also shows the values of two European options on the coupon bond, with strike 1, a notional of 100, and maturities at times 1 and 2, denoted E(t,T), for T = 1, 2. Finally, the tree shows the price of an American option on the coupon bond, with strike 1, a notional of 100, with maturity at time 2, denoted A(t,2).

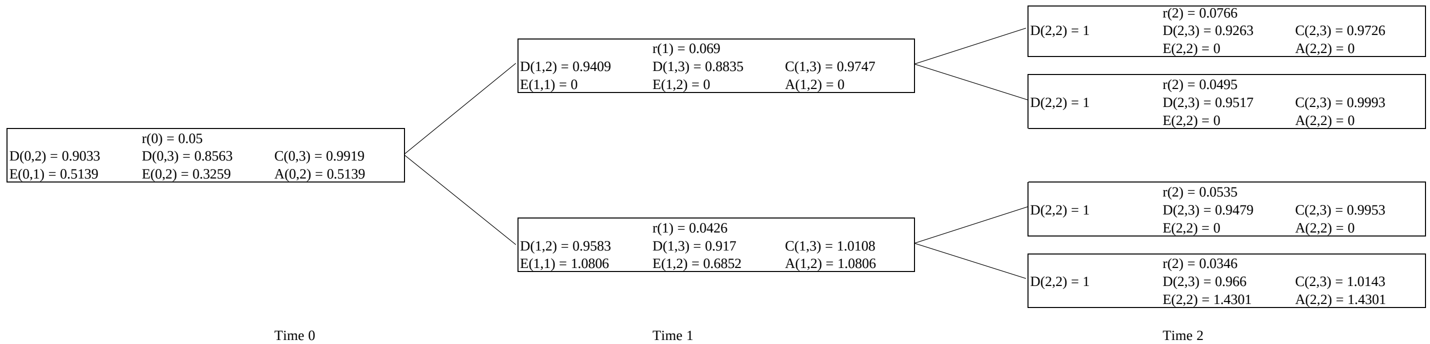

Figure 3

One-Factor Binomial Trees for Pricing an American Option on a Coupon Bond, Calibrated to Match the Bond Prices and European Option Prices at Time 1 of the Two-Factor Tree in Figure 1. The tree below shows four one-factor binomial models of the short rate, r, under the risk-neutral probability measure. The processes for the short rate are given by the recursion r(t+1) = r(t) exp(m(t)+s(t)u), where u can take the values -1 or 1 with probability 0.5. The values of m and s are chosen to fit the prices of bonds and European options observed in each of the nodes at time 1 in the two-factor tree of figure 1. The tree shows the values of the zero coupon bonds with maturities at times 2 and 3, where the price at time t of the discount bond with maturity T is denoted by D(t,T), for T = 2, 3, as well as the ex-coupon price of a coupon bond with maturity at time 3 and coupon rate 0.05, denoted C(t,3). The tree also shows the values of two European options on the coupon bond, with strike 1, a notional of 100, and maturities at times 1 and 2, denoted E(t,T), for T = 1, 2. Finally, the tree shows the price of an American option on the coupon bond, with strike 1, a notional of 100, with maturity at time 2, d

Recalibration at Node A

m(1) = 0.025550
s(1) = 0.13388

Time 1:
r(1) = 0.0649
D(1,2) = 0.9371 D(1,3) = 0.8763 C(1,3) = 0.9669
E(1,1) = 0 E(1,2) = 0 A(1,2) = 0

Time 2 (up):
r(2) = 0.0761
D(2,2) = 1 D(2,3) = 0.9267 C(2,3) = 0.973
E(2,2) = 0 A(2,2) = 0

Time 2 (down):
r(2) = 0.0583
D(2,2) = 1 D(2,3) = 0.94341 C(2,3) = 0.9906
E(2,2) = 0 A(2,2) = 0

Recalibration at Node B

m(1) = 0.020499
s(1) = 0.140655

Time 1:
r(1) = 0.0467
D(1,2) = 0.9544 D(1,3) = 0.9096 C(1,3) = 1.0028
E(1,1) = 0.2829 E(1,2) = 0.3554 A(1,2) = 0.3554

Time 2 (up):
r(2) = 0.0548
D(2,2) = 1 D(2,3) = 0.9467 C(2,3) = 0.994
E(2,2) = 0 A(2,2) = 0

Time 2 (down):
r(2) = 0.0414
D(2,2) = 1 D(2,3) = 0.9595 C(2,3) = 1.0074
E(2,2) = 0.7447 A(2,2) = 0.7447

Recalibration at Node C

m(1) = 0.013184
s(1) = 0.223266

Time 1:
r(1) = 0.0569
D(1,2) = 0.9447 D(1,3) = 0.8906 C(1,3) = 0.9824
E(1,1) = 0 E(1,2) = 0.1275 A(1,2) = 0.1275

Time 2 (up):
r(2) = 0.0720
D(2,2) = 1 D(2,3) = 0.9305 C(2,3) = 0.977
E(2,2) = 0 A(2,2) = 0

Time 2 (down):
r(2) = 0.0461
D(2,2) = 1 D(2,3) = 0.955 C(2,3) = 1.0027
E(2,2) = 0.2699 A(2,2) = 0.2699

Recalibration at Node D

m(1) = 0.024222
s(1) = 0.05713

Time 1:
r(1) = 0.0386
D(1,2) = 0.9621 D(1,3) = 0.9245 C(1,3) = 1.0188
E(1,1) = 1.8828 E(1,2) = 0.8876 A(1,2) = 1.8828

Time 2 (up):
r(2) = 0.0419
D(2,2) = 1 D(2,3) = 0.959 C(2,3) = 1.0069
E(2,2) = 0.6944 A(2,2) = 0.6944

Time 2 (down):
r(2) = 0.037
D(2,2) = 1 D(2,3) = 0.9633 C(2,3) = 1.0115
E(2,2) = 1.1506 A(2,2) = 1.1506

Time 1 Time 2