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General Equilibrium Stock Index Futures Prices: Theory and Empirical Evidence

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Abstract

We develop a closed-form general equilibrium model of stock index futures prices in a continuous-time economy with stochastic interest rates and market volatility. We show that futures prices implied by the model have very different properties from those of the cost of carry model. Using NYSE stock index futures data, we examine the restrictions imposed on futures prices by both the equilibrium and cost of carry models. Consistent with the equilibrium model, we find that stock index futures prices are related to market volatility and that their interest-rate sensitivity is a nonlinear function of contract maturity.

I. Introduction

The best-known model for pricing stock index futures is undoubtedly the cost of carry model. This model expresses the futures price in terms of the underlying stock index value, the risk-free interest rate, and the dividend yield for the index.[1] Its derivation relies on a simple no-arbitrage argument in which a trader replicates a futures position with spot positions in the stock and T-bill markets.

Despite its popularity, there are good reasons to consider alternatives to the cost of carry model. First, partial equilibrium or no-arbitrage models that assume the stock market is exogenous could fail to capture the dynamic interactions between spot and futures markets. This is important, given recent empirical evidence of this type of interaction—Kawaller, Koch, and Koch (1987), Ng (1987), and Stoll and Whaley (1990) document interdependence between stock index spot and futures prices. Second, futures and forward prices need

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[1]For a derivation of the cost of carry model as applied to stock index futures, see Cornell and French (1983) or Modest and Sundaresan (1983). The model is actually a forward (not futures) pricing model. To apply it to stock index futures, one must make the assumption that forward and futures prices are equal.

not be equal if interest rates are stochastic.[2] Third, there is empirical evidence of systematic pricing deviations from the cost of carry model. MacKinlay and Ramaswamy (1988) find that differences between actual stock index futures prices and theoretical prices based on the cost of carry model are serially correlated as well as path- and maturity-dependent. In addition, evidence in Resnick and Hennigar (1983), Kamara (1988), and Hemler (1988) suggests that pricing deviations from the cost of carry model are related to the level of interest rates or the volatility of the underlying security price in some financial futures markets.

Motivated by these considerations, this paper develops a closed-form general equilibrium model of stock index futures prices in a continuous-time production economy characterized by stochastic interest rates and fluctuating levels of market uncertainty. This model has the important advantage of allowing the stock index futures price to depend on fundamental economic variables such as the variance of returns on the market, instead of just the prices of traded assets. In addition, the general equilibrium approach allows for interactions between futures, cash, and credit markets; the risk-free rate, the stock index futures price, and the level of the stock index itself are determined endogenously as part of the equilibrium.

The general equilibrium model developed in this paper has many interesting and important implications for the behavior of stock index futures prices. For example, the model implies that stock index futures prices do not depend on preference-related market prices of risk even though futures prices are functions of nonprice variables. We show that even in the absence of dividends, the relation between the basis and contract maturity need not be monotonic. This result follows from the mean reversion in market volatility and the risk-free rate. Furthermore, percentage changes in the stock index futures price can be serially correlated and conditionally heteroskedastic. We derive a closed-form expression for the expected spot rate and show that futures prices are uniformly less than this expectation and that the difference is a function of both contract maturity and market volatility.

We also examine the relation between stock index forward and futures prices and show that these prices coincide when production returns are stationary. The simple functional form of the stock index futures price in the model implies that the logarithm of the futures/spot price ratio can be represented as a linear regression on both the risk-free rate and the volatility of the market. In contrast, the cost of carry model implies that the logarithm of the futures/spot price ratio depends on the risk-free rate only. These differences allow us to use a regression framework in testing the empirical implications of the equilibrium model.

Using NYSE stock index futures data for the 1983–1987 period, we find that market volatility has significant explanatory power for the futures/spot price ratio. In addition, we show that the slope coefficient for the interest rate is concave in maturity. These results are consistent with the equilibrium model, but not with the cost of carry model. We also estimate the parameters of the

[2]See Cox, Ingersoll, and Ross (1981), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981). For empirical evidence concerning the futures-forward price differential, see Rendleman and Carabini (1979), Cornell and Reinganum (1981), and French (1983).

equilibrium model and compare the model's out-of-sample performance to that of the cost of carry model.

The remainder of the paper is organized as follows. Section II uses the Cox, Ingersoll, and Ross (CIR (1985a), (1985b)) general equilibrium framework to derive a closed-form model of stock index futures prices in a simple production economy. Section III discusses the properties of the equilibrium stock index futures price. Section IV describes the empirical tests of the equilibrium and cost of carry models. Section V summarizes the results and presents concluding remarks.

II. A General Equilibrium Model of Stock Index Futures Prices

In this section, we use the general equilibrium framework of CIR (1985a), (1985b) to develop a closed-form model of stock index futures prices in an economy with both stochastic interest rates and market volatility.[3] In doing this, we make the following specific assumptions about preferences and production. There are a fixed number of identical individuals with time-additive preferences of the form,

$$E_t\left[\int_t^{\infty} e^{-\rho s}\ln(C(s))ds\right], \tag{1}$$

where $E_t[\,\cdot\,]$ is the conditional expectation operator and $C(s)$ represents time s consumption. All physical investment is performed by a single stochastic constant returns-to-scale production technology (firm), which produces a good that is either consumed or reinvested in production. We make the assumption of a single production technology since our focus is on stock index futures rather than futures contracts on specific firms. The returns on physical investment are governed by the stochastic differential equation,

$$\frac{dq}{q} = \mu X dt + \sigma\sqrt{Y}dZ_1, \tag{2}$$

where X and Y are economic state variables that induce random technological change, and Z_1 is a scalar Wiener process. The state variables X and Y are governed by the stochastic differential equations,

$$dX = (a + bX)dt + c\sqrt{X}dZ_2, \tag{3}$$

$$dY = (d + eY)dt + f\sqrt{Y}dZ_3, \tag{4}$$

[3]Cox, Ingersoll, and Ross (1981) present a general equilibrium model of discount bond futures prices in an economy with stochastic interest rates. Similarly, Ramaswamy and Sundaresan (1985) develop a no-arbitrage model of futures prices that incorporates stochastic interest rates. Bailey and Stulz (1989) examine the general equilibrium pricing of stock index options in a setting where interest rates and market volatility are stochastic but are driven by a single state variable and are perfectly correlated. This paper allows both interest rates and market volatility to be stochastic without requiring them to be perfectly correlated.

where a, $d > 0$, b, $e < 0$, and Z_2 and Z_3 are independent scalar Wiener processes. These production and state variable dynamics are a straightforward generalization of CIR (1985b), which assumes that the means and variances of production returns are proportional to a single state variable that follows a square root process. The production dynamics in (2) allow the mean and variance of production returns to depend on different state variables, both of which follow square root processes. Finally, we assume that there are perfectly competitive, continuous markets for risk-free borrowing and lending and for a variety of contingent claims including stock index futures contracts.

Following CIR (1985a), the representative investor's decision problem is equivalent to maximizing (1), subject to the budget constraint,

$$dW = W\frac{dq}{q} - Cdt, \tag{5}$$

where W denotes wealth, by selecting an optimal level of consumption and reinvesting unconsumed wealth in physical production. As in Merton (1971), the investor's derived utility of wealth function (value function) has the simple form,

$$J(W,X,Y,t) = \frac{e^{-\rho t}}{\rho}\ln(W) + G(X,Y,t), \tag{6}$$

and standard results can be used to show that optimal consumption is ρW. Substituting optimal consumption into (5) gives the following equilibrium dynamics for wealth

$$dW = (\mu X - \rho)Wdt + \sigma W\sqrt{Y}dZ_1. \tag{7}$$

Because all wealth is invested in production, W is also the value of the production technology (the value of the stock market). Thus, stock index dynamics are given endogenously by (7). Note that ρ now has the intuitive interpretation of a dividend yield; ρW is continuously withdrawn from the stock market and consumed. Together, W, X, and Y form a joint Markov process; the current values of W, X, and Y completely describe the state of the economy and the conditional distribution of future stock returns.

With this framework, we can now solve for the equilibrium stock index futures price. Rather than deriving the stock index futures price as a function of the unobservable state variables X and Y, however, we first make a simple change of variables that allows the futures price to be expressed in terms of three intuitive economic variables—the stock index value (W), the risk-free rate (r), and the variance of the market return (V). The advantage of this approach is that the stock index value and the risk-free rate can be directly observed while the variance can be empirically estimated. From Theorem 1 of CIR (1985a), the equilibrium risk-free interest rate is

$$r = \mu X - \sigma^2 Y. \tag{8}$$

Thus, the risk-free rate equals the expected return on the market minus the variance of the market return. This is intuitively reasonable since we would expect the risk-free rate to be less than the expected return on risky physical investment. Furthermore, we would expect the difference to be related to the riskiness of production returns.[4] In addition, from the dynamics in (7), the local variance of stock index return (V) is

$$V = \sigma^2 Y. \tag{9}$$

Since $\sigma^2 > 0$, this expression (in conjunction with the properties of Y in (4)) guarantees that V is nonnegative. Riskless production can occur if zero is an accessible value for Y. Together, (8) and (9) form a system of two linear equations in X and Y. Since μ, $\sigma^2 > 0$, the system is globally invertible and we can solve for X and Y in terms of r and V, obtaining

$$X = \frac{1}{\mu}(r+V), \tag{10}$$

$$Y = \frac{V}{\sigma^2}, \tag{11}$$

which allows the change in variables from X and Y to r and V. Applying Ito's Lemma to (8) and (9) gives the following dynamics for r and V,

$$dr = (\alpha - \beta r - \gamma V)dt + \eta\sqrt{r+V}dZ_2 - \xi\sqrt{V}dZ_3, \tag{12}$$

$$dV = (\delta - (\beta - \gamma)V)dt + \xi\sqrt{V}dZ_3, \tag{13}$$

where β, δ, η, $\xi > 0$, and $\gamma < \beta$. These parameters are related to the original parameters in (2), (3), and (4) by the following: $\alpha = \mu a - \sigma^2 d$, $\beta = -b$, $\gamma = e - b$, $\delta = \sigma^2 d$, $\eta^2 = \mu c^2$, and $\xi^2 = \sigma^2 f$. With this change of variables, we designate the equilibrium stock index futures price as $F(W, r, V, \tau)$, where τ represents the maturity (time until expiration) of the contract. From CIR (1985a), Theorem 3, and CIR (1981), Propositions 2 and 7, the stock index futures price satisfies the fundamental valuation equation,

$$\begin{aligned} \frac{W^2V}{2}F_{WW} &+ \left[\frac{\eta^2}{2}r + \left(\frac{\eta^2+\xi^2}{2}\right)V\right]F_{rr} + \frac{\xi^2 V}{2}F_{VV} \\ &+ \mathrm{Cov}(dW, dr)F_{Wr} + \mathrm{Cov}(dW, dV)F_{WV} - \xi^2 V F_{rV} + (r-\rho)WF_W \\ &+ \left[\alpha - \beta r - \gamma V - \frac{1}{W}\mathrm{Cov}(dW, dr)\right]F_r \\ &+ \left[\delta - (\beta-\gamma)V - \frac{1}{W}\mathrm{Cov}(dW, dV)\right]F_V = F_\tau, \end{aligned} \tag{14}$$

subject to $F(W, r, V, 0) = W$. The term $-rF$ does not appear in (14) because F is a futures price. As discussed by Black (1976) and CIR (1981) Proposition 2, an implicit "dividend" of rF is introduced by the resettlement feature of futures contracts, which cancels the $-rF$ term that otherwise would appear.[5]

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[4]Because of this property, the equilibrium real interest rate could be negative if the expected return on the market is close to zero.

[5]An alternative way of showing this is to note that CIR (1981), Proposition 2, implies that the futures price is the value of a contingent claim that pays

$$W \exp\left(\int_0^\tau r(s)ds\right)$$

Using the homogeneity of the initial condition in W, the closed-form solution to the three-dimensional parabolic partial differential equation in (14) can be obtained by the separation of variables method. The equilibrium futures price is

$$F(W,r,V,\tau) = We^{-\rho\tau}A(\tau)\exp(B(\tau)r + C(\tau)V), \tag{15}$$

where

$$A(\tau) = \left(\frac{2\phi\exp((\phi+\beta)\tau/2)}{(\beta+\phi)(\exp(\phi\tau)-1)+2\phi}\right)^{\frac{2(\alpha+\delta)}{\eta^2}} \times \left(\frac{2\psi\exp((\psi+\beta-\gamma)\tau/2)}{(\psi+\beta-\gamma)(\exp(\psi\tau)-1)+2\psi}\right)^{\frac{2\delta}{\xi^2}},$$

$$B(\tau) = \frac{2(\exp(\phi\tau)-1)}{(\beta+\phi)(\exp(\phi\tau)-1)+2\phi},$$

$$C(\tau) = B(\tau) + \frac{2(1-\exp(\psi\tau))}{(\psi+\beta-\gamma)(\exp(\psi\tau)-1)+2\psi},$$

$$\phi = \sqrt{\beta^2 - 2\eta^2}, \text{ and}$$

$$\psi = \sqrt{(\gamma-\beta)^2 + 2\xi^2}.$$

This expression shows that the equilibrium stock index futures price is an explicit function of W, r, V, and τ, and depends parametrically on the dividend-yield/time-preference parameter ρ and the parameters of the joint interest-rate/market-volatility process. Observe that the interest-rate/market-volatility parameters enter the stock index futures price only through the three terms $A(\tau)$, $B(\tau)$, and $C(\tau)$. This property will be particularly useful in the empirical tests in Section IV. Substituting $\tau = 0$ into (15) verifies that the equilibrium stock index futures price satisfies the expiration date condition $F(W,r,V,0) = W$. Note also that the futures price equals zero if the stock index value equals zero; $F(0,r,V,\tau) = 0$.

at maturity. This contingent claim satisfies (14) but with the $-rF$ term included on the right-hand side of the equation. However, from CIR (1981), (45) and (46), the value of this claim is just

$$E\left[W\exp\left(\int_0^\tau r(s)ds\right)\exp\left(-\int_0^\tau r(s)ds\right)\right] = E[W],$$

where the expectation is taken with respect to the risk-adjusted process (see CIR (1985a)). Because the two exponential terms in the expectation cancel out, it is straightforward to show (using Friedman (1975), Theorem 5.3) that the solution to (14) is equivalent to the futures price representation given in CIR (1981), (46) and (47). Thus, it is CIR (1981), Proposition 2 that allows us to drop the $-rF$ term in (14). We note also that (14) assumes that the futures contract is resettled continuously—actual futures contracts are resettled on a daily basis.

III. Properties of General Equilibrium Stock Index Futures Prices

In this section, we examine the properties of stock index futures prices implied by the general equilibrium model derived in the previous section. First, we show that the equilibrium stock index futures price does not depend on preference-related market prices of risk. We then derive comparative statics and describe the dynamic behavior of stock index futures prices in the equilibrium model. We also derive the relation between futures prices and the expected stock index level and show that the risk premium is a function of market volatility and contract maturity only. Finally, we examine the relation between forward and futures prices.

A. The Preference-Free Property

An important advantage of a general equilibrium approach over a no-arbitrage approach in valuing contingent claims is that a general equilibrium model can allow contingent claim prices to depend on variables that are not prices of traded assets. The cost of this generality, however, is often high since these nonprice variables cannot be hedged with traded securities. As shown by CIR (1985a), this means that risk premia related to the nonprice variables' effect on the marginal utility of wealth must generally appear in the fundamental valuation equation and the resulting contingent claim prices.

An important and surprising feature of the general equilibrium stock index futures model in (15) is that the futures price does not depend on any preference-dependent market price of risk terms (other than the time-preference parameter ρ), although the futures price is explicitly a function of nonprice variables. Intuitively, this is because the covariances of index returns with changes in r and V affect not only the expected returns on contingent claims, but also the variance of their returns. For stock index futures prices, these two effects exactly offset each other. Specifically, the market price of risk terms in (14), $-1/W\mathrm{Cov}(dW, dr)F_r$ and $-1/W\mathrm{Cov}(dW, dV)F_V$, are cancelled out by the second order terms, $\mathrm{Cov}(dW, dr)F_{Wr}$ and $\mathrm{Cov}(dW, dV)F_{WV}$, because $F_{Wr} = F_r/W$ and $F_{WV} = F_V/W$. An important advantage of this preference-free property is that market prices of risk do not need to be estimated to use the model.

This preference-free result depends on both the homogeneity of the futures price in W as well as the form of the stochastic process governing W, and is not shared by all general equilibrium models of futures prices. For example, CIR (1981) show that equilibrium discount bond futures prices depend on a market price of interest-rate risk parameter even though the representative investor is assumed to have logarithmic preferences. Ramaswamy and Sundaresan (1985) also present a model in which futures prices do not depend on market price of risk terms. In their model, however, the local expectations hypothesis is assumed to hold, which has the effect of eliminating the market price of interest-rate risk term.

B. Comparative Statics

From the expression in (15), it is easily shown that the general equilibrium stock index futures price F is a positive and monotone increasing function of the stock index level W. This follows because $A(\tau) > 0$, and the dynamics in (7) imply nonnegative W. Similarly, the stock index futures price is a decreasing function of the dividend yield; W is "discounted" by the dividend yield factor $e^{-\rho\tau}$ in (15), since $\rho > 0$.

The stock index futures price is a uniformly increasing function of the risk-free interest rate because $B(\tau) > 0$ for $\tau > 0$. This result is intuitive and is consistent with the well-known cost of carry model in which the interest rate can be viewed as a "carrying cost" that is added to the index value to obtain the futures price. As we will show later, however, the equilibrium and cost of carry models differ in their implications about the sensitivity of stock index futures prices to changes in the risk-free interest rate.

The remaining comparative statics are indeterminate. Partial differentiation reveals that the sign of F_V depends on the value of $C(\tau)$, which can be positive for some values of τ and negative for others. The intuition for this is that the current volatility of the index not only provides information about the distribution of the future index value, but also about future carrying costs since V affects the distribution of future interest rates. Thus, a change in market volatility can have complicated effects on the behavior of stock index futures prices that can differ across contract maturities. Similarly, the sign of F_τ is indeterminate even if $\rho = 0$. The reason for this is that both market volatility and the risk-free rate have mean-reverting components. Consequently, in some situations, the increase in carrying costs associated with an increase in τ can be more than offset by the mean-reversion-induced effects on the distribution of future stock index levels. These mean-reversion effects can be quite intricate. For example, when $\rho = 0$, the basis $F - W$ can change signs zero, one, or two times as τ ranges from 0 to ∞.

C. The Dynamic Behavior of Stock Index Futures Prices

The stochastic differential equation governing the dynamics of the equilibrium stock index futures price can be obtained by applying Ito's Lemma to the expression in (15). The instantaneous expected percentage change in F is given by

$$V + \mathrm{Cov}\left(\frac{dW}{W}, dr\right) B(\tau) + \mathrm{Cov}\left(\frac{dW}{W}, dV\right) C(\tau), \tag{16}$$

which can be either positive or negative depending on the values of the parameters, the correlation of stock index returns with changes in r and V, and the value of τ. The expected percentage change in F is directly related to the level of market volatility. As a result, expected percentage changes in the equilibrium stock index futures price are time varying. This property has important implications for the time series behavior of the stock index futures price. For example, it can be shown that the autoregressive dynamics for V given in (13) induce serial correlation in the expected percentage changes in F, which, in turn, induces serial correlation in the percentage changes in the stock index futures price.[6] Consequently, the equilibrium stock index futures model can imply serial correlation in successive futures price changes. This property is consistent with MacKinlay and Ramaswamy (1988) who find that the deviations of stock index futures prices from the cost of carry model are both maturity- and path-dependent; the equilibrium model implies that the deviations from the cost of carry model are related to both maturity and the level of market volatility.

The instantaneous variance of the percentage changes in F is given by the following expression,

$$(17)\quad B^2(\tau)\eta^2 r + [1 + B^2(\tau)(\eta^2 + \xi^2) - 2\xi^2 B(\tau)C(\tau) + C^2(\tau)\xi^2]V + B(\tau)\mathrm{Cov}\left(\frac{dW}{W}, dr\right) + C(\tau)\mathrm{Cov}\left(\frac{dW}{W}, dV\right),$$

which depends on both the levels of r and V. This demonstrates that percentage changes in the stock index futures price are conditionally heteroskedastic. The variance of percentage changes in F can be greater than or less than the variance of returns on the underlying stock index. If the covariance terms are nonnegative, however, then the variance of percentage changes in F is greater than the variance of index returns. Again, this property is consistent with the empirical findings of MacKinlay and Ramaswamy (1988). Partial differentiation indicates that this variance can be either increasing or decreasing as $\tau \rightarrow 0$ and may be increasing for some maturities and decreasing for others. If $\mathrm{Cov}(dW, dr) = \mathrm{Cov}(dW, dV) = 0$, however, the variance of percentage futures price changes decreases as $\tau \rightarrow 0$. These results contrast with Samuelson (1965) who argues that the variance of futures price changes should increase as $\tau \rightarrow 0$ because current information becomes more relevant in determining the future spot price.

D. The Relation between Futures Prices and Expected Index Values

Given the equilibrium stock index, interest rate, and market volatility dynamics in (7), (12), and (13), we can derive a closed-form representation of the expected stock index value and compare it to the corresponding futures price. This allows us to address one of the most important issues in futures pricing—the relation between the futures price and the expected spot price. This issue has been extensively debated since Keynes (1930), who theorized that hedgers would pay a premium to speculators for taking commodity price risk, resulting in futures prices that are less than the expected spot price (normal backwardation). Since Keynes, others such as Working (1948), Telser (1958), and Cootner (1960) have argued that futures prices may exceed expected spot prices (contango) or that futures prices are unbiased estimates of future spot prices (the unbiased expectations hypothesis).

[6]This follows immediately from the well-known property that time varying expected returns can induce a temporal covariance structure on returns. Longstaff (1989) presents a continuous-time example showing that returns can be positively and/or negatively serially correlated if expected returns follow an autoregressive process.

A conventional approach to finding the expected stock index value would be to solve the appropriate Fokker-Planck (Chapman-Kolmogorov) partial differential equation associated with the dynamics in (7), (12), and (13) to find the joint transitional density of W, r, and V, and then integrate to find the first conditional moment of W. However, since we are interested only in the expected stock index value, a more direct and tractable approach is possible. First, assume that dZ_1 and dZ_2 are uncorrelated (dZ_1 and dZ_3 may still be correlated).[7] This simplifies the task without affecting the stock index futures price since the futures price is independent of the correlation between dZ_1 and dZ_2. Next, note that the expected value of the stock index at the maturity of the futures contract, conditional on the current values of W, r, and V, $E_{W,r,V}[W(\tau)]$, is also the solution to the backward Fokker-Planck equation (with the appropriate initial condition) by Theorem 6.1 of Friedman (1975). Finally, apply a separation of variables approach to obtain the following solution for the conditional mean of $W(\tau)$,

$$(18) \qquad E_{W,r,V}[W(\tau)] = We^{-\rho\tau}D(\tau)\exp(B(\tau)r + B(\tau)V),$$

where $$D(\tau) = \left(\frac{2\phi\exp((\phi+\beta)\tau/2)}{(\beta+\phi)(\exp(\phi\tau)-1)+2\phi}\right)^{\frac{2(\alpha+\delta)}{\eta^2}},$$

and $B(\tau)$ is the same as in (15).

To compare the equilibrium stock index futures price directly to the expected future stock index value, let R denote the ratio of F to $E_{W,r,V}[W(\tau)]$. From (15) and (18),

$$(19) \qquad R = \frac{A(\tau)}{D(\tau)}\exp\left[(C(\tau) - B(\tau))V\right],$$

where $A(\tau)$ and $C(\tau)$ are defined in (15). From (19), R is a function of contract maturity and market volatility only; R is independent of W and r. Because $A(\tau) < D(\tau)$ and $C(\tau) < B(\tau)$, the ratio of the stock index futures price to the expected stock index value is less than one. Thus, (19) implies normal backwardation.[8] The intuition behind this result is that the stock index futures prices can be represented as the expected stock index value, given the "risk-adjusted" dynamics of W, while the actual expected stock index value is based on the original dynamics; the upward drift implied by the "risk-adjusted" drift is less than that implied by the original dynamics.[9]

[7]This assumption implies that the unexpected return on the market is uncorrelated with changes in the expected return on the market. Admittedly, this assumption is somewhat restrictive. However, similar results can be obtained (although not in closed-form) by requiring that dZ_1 and dZ_2 be negatively correlated, rather than uncorrelated.

[8]Hemler and Longstaff (1990) test for normal backwardation in S&P 500 stock index futures prices during the 1983–1989 period. When October 1987 is excluded, we find significant evidence of normal backwardation for contracts with maturities of two and three months.

[9]From the stock index dynamics in (7), the instantaneous expected return for the index is $\mu X - \rho$. However, from (14), the implied instantaneous expected return for the index is $r - \rho$, which is less than or equal to $\mu X - \rho$ from (18). The intuition for why the futures price can be viewed as the expectation of the future stock index value with respect to the "risk-adjusted" process follows

Since R can be viewed as the risk premium implicit in the stock index futures price, it is important to determine how the risk premium is related to market volatility and contract maturity. Partial differentiation shows that R is a decreasing function of stock market volatility. Similarly, the risk premium is a decreasing function of contract maturity. Observe that the dependence on V implies that the risk premium in the stock index futures price is time varying.

E. The Relation between Forward and Futures Prices

Recall that interest rates are stochastic in this general equilibrium setting, which implies that stock index futures and forward prices need not be equal (see CIR (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981)). The equilibrium futures price is given in (15). The corresponding forward price is easily determined using the no-arbitrage cost of carry model (or equivalently, Proposition 1 of CIR (1981)) and can be written as $We^{-\rho\tau}/P(\tau)$, where $P(\tau)$ is the value of a discount bond with maturity τ. From CIR (1985a), (1985b), $P(\tau)$ also satisfies a partial differential equation similar to (14), subject to the condition $P(0) = 1$. Although we are unable to obtain a closed-form expression for $P(\tau)$, it is straightforward to show that the equilibrium discount bond price depends on the variables r, V, τ, and the covariance terms $\mathrm{Cov}(dW, dr)$ and $\mathrm{Cov}(dW, dV)$.[10] Thus, discount bonds (and, therefore, forward prices) do not have the preference-free property of the futures price.

One important implication of the cost of carry model is that the forward price depends on V through $P(\tau)$. Thus, given $P(\tau)$ or the τ-maturity yield, the current value of V should provide no additional information about the forward price. In contrast, V does provide incremental information about the futures price. Hence, the cost of carry model and the general equilibrium model can be distinguished by their implications for the relation between stock index futures prices and market volatility. This distinction will form the basis for the empirical tests of the cost of carry and general equilibrium models in the next section.

In the special case when production is not subject to technological change, the forward and futures prices coincide. In this situation, both the risk-free rate and the variance of the market's returns are constant, which implies $\alpha = \beta = \gamma = \delta = \eta = \xi = 0$. To see this, substitute these parameter restrictions in (14) to give the following partial differential equation for the futures price $F(W, \tau)$,[11]

$$W^2 V F_{WW} + (r - \rho) W F_W = F_\tau, \tag{20}$$

immediately from Theorem 5.3 of Friedman (1975) in which the solution to (14) can be represented as an expectation of the expiration date condition (the future stock index value), with respect to the joint stochastic process defined by the coefficients of the first- and second-order terms in the partial differential equation.

[10] The partial differential equation satisfied by $P(\tau)$ is obtained from (14) by adding a $-rP$ term to the left-hand side and dropping all of the partial derivatives that depend on W. Because these derivatives are dropped, however, the covariance terms $\mathrm{Cov}(dW, dr)$ and $\mathrm{Cov}(dW, dV)$ in the first-order terms of the partial differential equation are not cancelled out. Thus, the equilibrium discount bond price depends on the market prices of interest rate and market volatility risk.

[11] We cannot simply impose this parameter restriction on the closed-form expression in (15) since this results in mathematically undefined terms. The intuition for this is that the parameter restriction $\alpha = \beta = \gamma = \delta = \eta = \xi = 0$ essentially changes the valuation equation from a second-order partial differential equation to a first-order partial differential equation, which, in turn, alters the functional form of the solution.

subject to the initial condition $F(W,0) = W$. Solving (20) gives

$$F(W,\tau) = We^{(r-\rho)\tau}, \quad (21)$$

which is just the cost of carry model for forward prices since $P(\tau) = e^{-r\tau}$ when r is constant. This result is consistent with CIR (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981), who show that forward and futures prices are equal when the interest rate is nonstochastic. In the equilibrium stock index futures model, the interest rate is nonstochastic if, and only if, the state variables X and Y are nonstochastic. If X and Y are nonstochastic, however, then V is also nonstochastic. Thus, the absence of technological change in the equilibrium model implies the equality of forward and futures prices.

In the general case where r and V are stochastic, little can be said about the relation between forward and futures prices. This is because the forward price depends on the market price of risk terms $\mathrm{Cov}(dW, dr)$ and $\mathrm{Cov}(dW, dV)$ (through its dependence on $P(\tau)$), while the futures price does not. In the special case where these covariances are zero, however, the stock index futures price exceeds the forward price. This follows from CIR (1981) Proposition 9 because percentage changes in the forward and discount bond prices are instantaneously negatively correlated if $\mathrm{Cov}(dW, dr) = \mathrm{Cov}(dW, dV) = 0$.

IV. Empirical Results

A. The Data

The data include futures prices and index values for the New York Futures Exchange (NYFE) contract based on the New York Stock Exchange Composite Index. The S&P 500 contract traded on the Chicago Mercantile Exchange (CME) might appear to be a more natural choice due to its larger trading volume and open interest. The problem with using the S&P 500 Index, however, is the relative unavailability of accurate monthly and daily dividend data. To our knowledge, such data are not publicly available. Researchers (e.g., MacKinlay and Ramaswamy (1988)) frequently handle this problem by using dividend data from the Center for Research in Security Prices (CRSP). However, the monthly (respectively, daily) CRSP data correspond to the NYSE (respectively, NYSE/AMEX) portfolio. Because the NYSE and NYSE/AMEX portfolios contain a higher proportion of small firms than the S&P 500 portfolio, we expect their dividend yields to be different from the S&P 500 dividend yield. Consequently, rather than mixing index values and futures prices for the S&P 500 with dividend data for the NYSE or NYSE/AMEX portfolio, we use the corresponding data for the NYSE portfolio alone.

The futures price and index value quotations are from *The Wall Street Journal*. Trading in the NYSE futures contract began on May 6, 1982. Because early trading volume was often light, we use data beginning in 1983 to allow the market time to become seasoned and to minimize problems with nonsynchronous data. Moreover, we use monthly observations because one month is the smallest time interval for which we can obtain exact dividend data for the NYSE portfolio. Observations correspond to the last business day for each month from January 1983 through November 1987. All futures prices are for the nearby contract, which implies that the corresponding times to expiration are approximately one, two, or three months. Before December 1984, contracts expired on the last business day of the delivery month. Since December 1984, contracts have expired on the third Friday of the delivery month. This change leads to slight variations in the actual number of days to expiration for the one-, two-, and three-month maturity contracts. However, this should have no material effect on our empirical tests.

To measure market volatility, we estimate the variance of monthly stock market returns using the procedure of French, Schwert, and Stambaugh (1987).[12] That is, we estimate the monthly variance as the sum of the squared daily returns plus twice the sum of the product of adjacent returns over all days within the respective month. The variance estimates are annualized by multiplying them by 12. This procedure assumes that the market's variance is constant over a month. Thus, the resulting estimates are only discrete approximations of the value of V that appears in the general equilibrium model. The likely effect of this, however, is to bias the tests against finding a relation between futures prices and volatility. The daily value-weighted returns used in this procedure are from CRSP. These returns correspond to the combined NYSE/AMEX value-weighted portfolio rather than the NYSE portfolio alone. Although we would prefer using volatility estimates based solely on the NYSE portfolio, we believe that differences between these estimates are minor.

To calculate theoretical futures prices, we also use interest rate and dividend yield data. For the risk-free interest rate, we use the average of Treasury-bill bid and ask discount yield quotations as published in *The Wall Street Journal* for the maturity corresponding to the futures contract. We note that the risk-free rate for the maturity corresponding to the futures contract is not precisely the same as the instantaneous risk-free rate in the equilibrium stock index futures model. By using the risk-free rate corresponding to the futures contract, however, we are able to compare the regression results for the equilibrium model directly with those for the cost of carry model. We estimate monthly dividend yields using CRSP monthly value-weighted returns (including and excluding dividends) for the NYSE portfolio. In those cases where a contract expired before the end of a month, we use the total monthly dividend yield multiplied by the percentage of the month that the contract traded.

Table 1 presents descriptive statistics for the NYSE Composite Index along with other variables used in the regression analysis. The most notable feature of the data is the degree of autocorrelation present. As expected, the index W_t exhibits time-series behavior similar to a first-order autoregressive process having an autoregressive parameter close to unity. The interest rate r_t also behaves like a first-order autoregressive process; its autocorrelation coefficients are large and decay slowly. On the other hand, only the third autocorrelation coefficient for the logarithm of the dividend-adjusted futures/spot price ratio $\ln(F_t e^{\rho\tau}/W_t)$

[12]Other measures of market volatility might be used. For example, implied volatilities could be calculated by applying a variant of the Black-Scholes (1973) formula to NYSE Index options. However, this presents theoretical problems. In particular, our model implies that the risk-free rate and market volatility are stochastic while the Black-Scholes model assumes that they are constant.

is large. This is because consecutive observations of this variable reflect the interest rate multiplied by different times to maturity, where the times to maturity cycle from three months to two months to one month. The large autocorrelation coefficients suggest that problems with autocorrelated residuals might arise in the forthcoming regression analysis. Such problems never materialize, however, as long as the interest rate variable is included as a regressor.

TABLE 1

Summary Statistics[a] for the NYSE Composite Index, the Logarithm of the Dividend-Adjusted Futures/Spot Price Ratio, Treasury Bill Yields, and the Variance of the Market's Return, Using Monthly Data for the January 1983 to November 1987 Period

Variable	W_t	$\ln(F_t e^{\rho\tau}/W_t)$	r_t	V_t
Sample Size	59	59	59	59
Mean	118.274	0.012	0.0711	0.0337
Median	109.390	0.010	0.0710	0.1058
Minimum	83.750	−0.003	0.0341	0.0058
Maximum	184.450	0.034	0.1020	0.8214
Standard Deviation	28.532	0.009	0.0159	0.1053
1st Order Autocorrelation	0.952	0.104	0.814	0.060
2nd Order Autocorrelation	0.892	0.079	0.760	0.008
3rd Order Autocorrelation	0.822	0.733	0.772	−0.008

[a] Observations correspond to the last business day for each month. Futures prices (F_t) and index values (W_t) are for the NYFE contract based on the NYSE Composite Index. Futures prices are for the nearby contract. Interest rates (r_t) are based on Treasury bill discount yield quotations from *The Wall Street Journal.* Dividend yields (ρ) are estimated using CRSP monthly value-weighted returns for the NYSE portfolio. The variance of the market's return (V_t) is estimated by applying the French, Schwert, and Stambaugh (1987) technique to CRSP daily value-weighted returns to obtain variance estimates for each month of the sample period. The variance estimates are then annualized by multiplying them by 12.

B. Regression Analysis

To test the equilibrium stock index futures model, we first derive the implications of the model for the natural logarithm of the dividend-adjusted futures/spot price ratio $\ln(F_t e^{\rho\tau}/W_t)$, which we denote $L_{\tau t}$. This choice of dependent variable is particularly useful because the model's implications for $L_{\tau t}$ have an intuitive interpretation as simple restrictions on the parameters of a linear regression equation. In addition, the cost of carry model can be nested within the regression model. Thus, we can test the restrictions imposed by the cost of carry model within the same framework.

The exponential relation for the futures price in the equilibrium model implies that $L_{\tau t}$ has a simple linear form. Specifically, taking the natural logarithm of the closed-form expression for the equilibrium stock index futures price in (15) and rearranging terms gives the following regression equation,

$$L_{\tau t} = \alpha_\tau + \beta_\tau r_t + \gamma_\tau V_t + \epsilon_t, \tag{22}$$

where the regression coefficients α_τ, β_τ, and γ_τ are related to $A(\tau)$, $B(\tau)$, and $C(\tau)$, and the residual term $\epsilon_{\tau t}$ is included to reflect the possibility of measurement error in the $L_{\tau t}$ terms due to the F_t, W_t, and ρ estimates. Measurement errors in F_t, W_t, and ρ could potentially be due to nonsynchronous price data, bid/ask spreads, nontransaction prices in the data source, or intramonth interpolation of the dividend yields. Incorporating these variables into the dependent variable $L_{\tau t}$ allows us to avoid the errors in variables problem that might occur if these variables were to appear as independent variables in the regression. The residuals ϵ_t are assumed to be independently and identically distributed normal variates with mean zero. In a similar fashion, taking the natural logarithm of the cost of carry forward price $We^{\rho\tau}/P(\tau)$ (where $P(\tau)$ equals $e^{-r\tau}$ since r is now the τ-maturity yield), and rearranging also leads to the regression equation in (22). The regression coefficients in the cost of carry model, however, have a different interpretation than in the equilibrium stock index futures model. Thus, although both models imply a linear regression equation for the ratio $L_{\tau t}$, the models can be distinguished from each other by their respective implications for the coefficients α_τ, β_τ, and γ_τ.

The restrictions imposed by the cost of carry model on the regression coefficients in (22) are derived from the forward price. For notational simplicity, let α_1 denote the intercept for the one-month maturity contract, α_2, the intercept for the two-month maturity contract, and α_3, the intercept for the three-month maturity contract, and similarly for the other regression coefficients. The coefficient restrictions implied by the cost of carry model are

$$\alpha_i = 0, \quad i = 1, 2, 3, \tag{23}$$

$$\beta_i = \tau, \quad i = 1, 2, 3, \tag{24}$$

$$\gamma_i = 0, \quad i = 1, 2, 3. \tag{25}$$

The restrictions on α_i follow immediately from the functional form of the cost of carry model. The restrictions on γ_i arise because the cost of carry model implies that market volatility should not have explanatory power for $L_{\tau t}$ after including the interest rate in the regression. Finally, the cost of carry model implies that the coefficient of the interest-rate term should equal the contract maturity, which leads to the restriction on the β_i coefficients.

The implications of the equilibrium model for the regression coefficients can be derived directly from the expressions for $A(\tau)$, $B(\tau)$, and $C(\tau)$ in (15). The resulting restrictions on the regression coefficients are

$$0 < \beta_1 < \beta_2 < \beta_3, \tag{26}$$

$$2\beta_2 \geq \beta_1 + \beta_3, \tag{27}$$

$$\gamma_i < \beta_i, \quad i = 1, 2, 3. \tag{28}$$

The inequality restrictions on the β_i coefficients in (26) arise because $B(\tau)$ is a positive and monotone increasing function of τ in the equilibrium stock index futures model (but need not equal τ). The second restriction follows because $B(\tau)$ is a concave function of τ. (Equality in (27) holds if, and only if, the cost of carry model holds.) Finally, $C(\tau)$ is uniformly less than $B(\tau)$ in the equilibrium stock index futures model. Thus, each of the γ_i coefficients should be less than the β_i coefficient for the contract with the same maturity. These restrictions, along with those of the cost of carry model, can be examined directly in a linear regression framework. Observe that the coefficient restrictions imposed by the cost of carry model are stronger than those obtained for the general equilibrium model. This is to be expected. It reflects the fact that the cost of carry model can be nested as a special case of the general equilibrium model.

To test these restrictions, we estimate the regression equation using data for the period from January 1983 through November 1987. The estimation procedure is the same as for ordinary least squares, except that the standard errors for the estimated regression coefficients are adjusted using the White (1980) correction for conditional heteroskedasticity. Because of the limited amount of data for individual maturities, the data for the one-, two-, and three-month contracts are pooled in the regressions. However, we use dummy variables in order to allow the α, β, and γ coefficients to vary across the different maturities. The estimated regression equation is

$$L_{\tau t} = \sum_{i=1}^{3} \alpha_i D_i + \sum_{i=1}^{3} \beta_i D_i r_t + \sum_{i=1}^{3} \gamma_i D_i V_t + \epsilon_t, \tag{29}$$

where D_i are dummy variables that take on the value of 1 if the dependent variable corresponds to an i-month to maturity contract and the value 0 otherwise.[13] Thus, the regression results in the nine parameter estimates α_i, β_i, and γ_i for $i = 1, 2, 3$.[14]

Table 2 reports the regression results. Because the October 1987 market variance estimate is much larger than the variance estimates for the other months in the sample, the regressions are estimated both with and without the October 1987 observation in order to guard against the possibility that the inferences are driven by an influential observation. Because of the dummy variables in the regression, including the October 1987 observation in the sample affects only the coefficient estimates for the two-month maturity futures contract.

As shown in Table 2, most of the empirical implications of the cost of carry model are easily rejected. For example, both $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are statistically different from zero in each of the two regressions; the t-statistic for $\hat{\alpha}_1$ is -4.09 and the t-statistic for $\hat{\alpha}_2$ is -2.02 or -3.28 depending on whether the October 1987 observation is included. Recall that the cost of carry model implies that

[13] Note that this regression does not include an intercept term since three dummy variables are used; including an intercept term would result in perfect multicollinearity. It is easily shown that this regression is equivalent to the more traditional form of a dummy variable regression in which there are only two dummy variables (for the two- and three-month maturities). With only two dummy variables, the parameters for the two- and three-month maturity contracts are obtained by adding the coefficients for the dummy variables to those for the one-month maturity contracts; this regression gives the same coefficients for the two- and three-month maturity contracts directly. See Maddala (1977).

[14] Fama and French (1987) estimate a similar regression in which the standardized basis is regressed on $r\tau$ (using our notation). Our tests differ from those of Fama and French because we include the variance of the market in the regressions. This enables us to test the cost of carry model against the specified alternative of the equilibrium model, which allows the tests to have greater power against the cost of carry model.

TABLE 2

Regressions[a] of the Natural Logarithm of the Dividend-Adjusted NYSE Stock Index Futures/Spot Price Ratio on the Risk-Free Interest Rate and the Variance of the Market's Return Using Monthly Data for the January 1983 to November 1987 Period

$$L_{\tau t} = \sum_{i=1}^{3} \alpha_i D_i + \sum_{i=1}^{3} \beta_i D_i r_t + \sum_{i=1}^{3} \gamma_i D_i V_t + \epsilon_t$$

Period	α_1	α_2	α_3	β_1	β_2	β_3	γ_1	γ_2	γ_3	Adjusted R^2	dW	N
Excluding 10–87	−0.00878	−0.01383	−0.00619	0.16050	0.37395	0.36724	0.09728	0.02087	−0.19633	0.702	1.68	58
	(−4.09)[b]	(−2.02)	(−1.03)	(5.57)	(5.06)	(4.99)	(3.08)	(0.22)	(−2.86)			
Including 10–87	−0.00878	−0.01436	−0.00619	0.16050	0.37801	0.36724	0.09728	0.03222	−0.19633	0.717	1.70	59
	(−4.09)	(−3.28)	(−1.03)	(5.57)	(6.50)	(4.99)	(3.08)	(13.61)	(−2.86)			

[a] $L_{\tau t}$ is the ratio $F_t e^{\rho\tau}/W_t$, where F_t is the nearby NYSE stock index futures price (one-, two-, or three-month maturity), W_t is the level of the NYSE index, ρ is the average dividend yield over the life of the contract, and τ is the maturity of the contract measured in years (360 days). D_i is a dummy variable that takes values 1 if the nearby futures contract has a maturity of i months and zero otherwise. r_t is the annualized Treasury bill rate for maturity τ, and V_t is the annualized volatility of the CRSP NYSE/AMEX index computed using daily returns during month t. All prices used are closing prices for the end of the month.

[b] t-statistics in parentheses are based on the White (1980) heteroskedasticity consistent estimate of the covariance matrix of the parameters.

all of the α_i coefficients should be zero. Table 2 also shows that the variance of the market's return has significant explanatory power for L_{Tt} for the one- and three-month contract maturities when October 1987 is excluded; the t-statistics for $\hat{\gamma}_1$ and $\hat{\gamma}_3$ are 3.08 and −2.86, respectively. When October 1987 is included, $\hat{\gamma}_2$ is also significant with a t-statistic of 13.61. Note that the point estimate for $\hat{\gamma}_2$ is essentially unaffected by whether or not October 1987 is included. This suggests that the October 1987 observation, while extreme, is consistent with the other months. Again, the significance of market volatility in the tests provides evidence against the cost of carry model, which implies that the variance should not have significant explanatory power after including the risk-free rate in the regression. Finally, the cost of carry model implies that the β_i coefficients should equal the maturity of the contract. The average maturities for the one-, two-, and three-month contracts during the sample period are 0.064, 0.148, and 0.232 years, respectively. $\hat{\beta}_1$ is 3.34 standard deviations larger than 0.064, $\hat{\beta}_2$ is either 3.06 or 3.96 standard deviations larger than 0.148, and $\hat{\beta}_3$ is 1.84 standard deviations larger than 0.232. Thus, all three of the estimates of β_i are larger than implied by the cost of carry model and two are significantly larger at the 0.05 level.[15]

Although the cost of carry model's implications are rejected, the regression results are consistent with the empirical implications of the equilibrium stock index futures model. As mentioned, market volatility has significant explanatory power for two (or three, if the October 1987 observation is included) of the three maturities examined. This result is important because it provides a possible explanation for the perceived failure of the cost of carry model in turbulent and volatile markets. As implied by the equilibrium model, each of the $\hat{\beta}_i$ coefficients is positive. Although the $\hat{\beta}_i$ coefficients are not strictly monotone increasing ($\hat{\beta}_3$ is slightly less than $\hat{\beta}_2$), the $\hat{\beta}_i$ coefficients are concave in maturity as implied by the model. Finally, each of the $\hat{\gamma}_i$ coefficients is less than the corresponding $\hat{\beta}_i$ coefficient, and the $\hat{\gamma}_2$ and $\hat{\gamma}_3$ coefficients are significantly less at the 0.05 level.

Table 2 also shows that the $\hat{\alpha}_i$ coefficients are all negative. The $\hat{\alpha}_i$ coefficients are initially a decreasing function of contract maturity, but then become an increasing function. Differentiating $A(\tau)$ indicates that this pattern for the $\hat{\alpha}_i$ coefficients is consistent with the equilibrium model. In addition, the other $\hat{\gamma}_i$ coefficients are monotone decreasing in maturity. Again, differentiating $C(\tau)$ shows that this pattern is consistent with the equilibrium stock index futures model.

C. A Comparison of the Equilibrium and Cost of Carry Models

To compare the performance of the equilibrium model to that of the cost of carry model in explaining the level of actual stock index futures prices, we

[15]Interestingly, (8) implies that the short-term interest rate and the level of market volatility are negatively correlated—the correlation between r_t and V_t during the sample period is −0.25. Although this correlation is not large, we also estimated the regressions using only one of the two independent variables at a time in order to provide a diagnostic check for multicollinearity. The regression results obtained were comparable to those reported.

TABLE 3

Summary Statistics for the Pricing Errors[a] of the Cost of Carry and Equilibrium Models for the Nearby NYSE Stock Index Futures Price during the January 1986 to November 1987 period (23 monthly observations)

Statistic[b]	Cost of Carry Including 10-87	Equilibrium Including 10-87	Cost of Carry Excluding 10-87	Equilibrium Excluding 10-87
RMSE	0.9439	0.8687	0.6572	0.8443
Mean	0.0611	−0.0534	0.2146	0.0031
Standard Deviation	0.9419	0.8671	0.6212	0.8442
Correlation with r	0.170	0.254	−0.049	0.189
Correlation with V	−0.758	−0.336	0.036	−0.351
Correlation with F	−0.064	−0.154	−0.182	−0.184
Correlation with τ	0.041	−0.115	0.068	−0.120
1st Order Autocorrelation	0.242	0.290	−0.050	0.200
2nd Order Autocorrelation	0.171	−0.023	0.140	−0.112
3rd Order Autocorrelation	0.220	−0.003	0.198	−0.065

[a] Pricing errors are the difference between the actual nearby NYSE stock index futures price and the respective model price. The parameters used in the equilibrium model are computed from the regression of the natural logarithm of the dividend-adjusted futures/spot price ratio on the risk-free rate and the market's volatility where the regressions use data from 1-83 up to, but not including, the month for which the parameters are used.

[b] RMSE is the root mean squared error, r is the risk-free interest rate for the maturity of the nearby futures contract, V is the annualized variance of the CRSP NYSE/AMEX index returns computed monthly using daily data, and τ is the maturity of the nearby futures contract in years (360 days).

first need to obtain estimates of the input parameters for the equilibrium model, $A(\tau)$, $B(\tau)$, and $C(\tau)$. As before, these parameters can be directly estimated from regressions of $L_{\tau t}$ on the interest rate and market volatility. In estimating the parameters as regression coefficients, our strategy is to use the data from January 1983 up to month t to obtain the estimates of $A(t+1)$, $B(t+1)$, and $C(t+1)$ that are used in the equilibrium model. This approach has the advantage of allowing parameters to be estimated using only ex ante data while also using as much information as possible in the regressions.[16] In order to allow a reasonable parameter information period while still retaining sufficient data for an out-of-sample test, the first set of parameters is estimated using the 36-month period from January 1983 to December 1985. Consequently, the out-of-sample tests are based on the 23-month period from January 1986 to November 1987.

Table 3 presents summary statistics for pricing errors of both the cost of carry and equilibrium stock index futures models for the January 1986 to November 1987 period. As before, we report the results both with and without the October 1987 observation. When October 1987 is included, the equilibrium model results in a smaller root mean squared error than the cost of carry model;

[16] We also tried estimating the parameters $A(\tau)$, $B(\tau)$, and $C(\tau)$ as implied values using the preceding three futures prices for contracts with maturity τ. However, this procedure did not always provide reliable parameter values because small errors in the futures and stock index values can be magnified in the solution procedure, which requires inverting a Jacobian matrix. The use of a regression approach leads to more stable parameter values since the effects of individual errors in the price data are smoothed or dampened out.

the root mean squared error for the equilibrium model is 0.869 while the root mean squared error for the cost of carry model is 0.941. When October 1987 is excluded, the cost of carry model results in a lower root mean squared error. The reasons for the difference in results is that the October 1987 pricing error for the cost of carry model is very large—when October 1987 is dropped from the sample, the cost of carry model's performance improves appreciably. Note, however, that the pricing errors for October 1987 are computed using data from the end of the month—the extreme pricing error for the cost of carry model in October 1987 occurred almost two weeks after the stock market crash. Both models provide relatively unbiased estimates of the stock index futures price when all months are included. The bias for the cost of carry model is 0.061 index points (standard error 0.197), while the bias for the equilibrium model is −0.053 (standard error 0.183).

Table 3 also presents univariate correlations of the pricing errors with the risk-free interest rate, the market's volatility, contract maturity, and the level of the futures price. When October 1987 is included, only the correlation between the pricing errors for the cost of carry model and the market volatility is significant. Table 3 also reports the first three autocorrelations for the two models' pricing errors.

V. Summary and Conclusions

This paper develops a general equilibrium model of stock index futures prices in a continuous-time economy with stochastic interest rates and market volatility. The implications of this general equilibrium model for stock index futures prices are generally quite different from those of the cost of carry model and are testable using regression analysis. When the natural logarithm of the dividend-adjusted futures/spot price ratio is regressed on market volatility and interest rates, we find that market volatility has significant explanatory power. Moreover, the regression coefficients for the interest-rate variable are concave in the time to expiration. These results are consistent with the general equilibrium model, but not the cost of carry model.

These results are provocative in terms of their implications for hedging behavior. If one accepts the general equilibrium model, then market volatility plays an important role in determining the stock index futures price. However, traditional strategies of hedging changes in the underlying stock index generally do not incorporate information about market volatility. Hence, one implication of our results is that these strategies cannot provide a perfect hedge. It would be interesting to determine whether hedging strategies based on this general equilibrium model systematically outperform traditional strategies.

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