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Pricing Options with Extendible Maturities: Analysis and Applications

FRANCIS A. LONGSTAFF*

ABSTRACT

Many common types of financial contracts incorporate options with extendible maturities. This paper derives closed-form expressions for options that can be extended by the optionholder and presents a number of applications including the valuation of American options with stochastic dividends, junk bonds, and shared-equity mortgages. We also derive closed-form expressions for writer-extendible options and discuss the writer's economic incentives for extending an out-of-the-money option. We apply these results to show that corporate debtholders have a strong incentive to extend the maturity of defaulting debt if there are liquidation costs. We model and solve the debtholders' optimal extension problem and show that the possibility of an extension can induce shareholders in highly levered firms to accept negative NPV projects.

MANY COMMON TYPES OF financial contracts and contingent claims incorporate options with extendible maturities. For example, a growing number of firms (in many cases, firms involved in leveraged buyouts) are issuing bonds with maturities that can be extended at the firm's option. Options on real estate often allow the optionholder to extend the expiration date by paying an additional fixed amount to the option writer. Corporate warrants frequently give the issuing firm the right to unilaterally extend the life of the warrants—a right that is often exercised. In general, any financial contract that could involve a rescheduling of payments, a renegotiation of terms, an early call or exercise provision, or some similar type of flexibility over the timing of cash flows could be viewed as including an option with an extendible maturity.[1]

This paper derives valuation expressions for extendible options, examines the analytical properties of these prices, and presents a wide variety of applications and examples. In this analysis, we distinguish between options that can be extended by the optionholder and options that are extended by the option writer, since they differ fundamentally in their pricing implications. Focusing on the

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[1] Brennan and Schwartz (1977) and Ananthanarayanan and Schwartz (1980) study several related contingent claims—retractable and extendible bonds. Their emphasis, however, is quite different from ours since the fundamental state variable in their analysis is the risk-free interest rate instead of an underlying asset price. In addition, they focus on the valuation of bonds, while this paper addresses the pricing of extendible options. Consequently, although complementary, the valuation results in this paper are fundamentally different from theirs.

first type, we show that the optionholder's payoff function can be expressed as the maximum of three different cash flows. We characterize the optimal extension policy for the optionholder and use these results to derive closed-form expressions for the values of extendible calls and puts. We also derive comparative statics results, obtain rational bounds for extendible option prices, and show that a number of other contingent claims, such as compound options (Geske (1979a)), are special cases of extendible options. We then present a variety of examples of contingent claims that can be valued using these results including real estate options, junk bonds with extendible maturities, warrants with exercise price changes, American calls on stocks that pay stochastic dividends, and shared-equity mortgages.

Turning to writer-extendible options, we address the important economic issue of why an option writer might choose to extend the life of an option. We show that, if an option writer faces a penalty when the option expires out-of-the-money (such as liquidation costs or income taxes on the original premium), the writer can have a strong incentive to extend the option's life. We derive closed-form expressions for the values of simple calls and puts that are extended by the writer for a given period if out-of-the-money at the initial expiration date. We show that the properties of these options are very different from those of conventional options with nonstochastic maturities. For example, these options need not be monotonic functions of the underlying asset price or of the volatility of the asset's returns. The intuition for these results is that an optionholder may sometimes prefer to have a second chance to exercise an option instead of a slightly-in-the-money expiration. Consequently, an increase in the probability of an in-the-money expiration is not always favorable for optionholders.

Finally, we examine the pricing of options that are extendible by the writer for a period of time that depends on the underlying asset price at the initial expiration date. Focusing on the specific example of stock in a risky levered corporation where there are liquidation costs, we find that a lender always has an incentive to extend the maturity of the debt in the event of a default. We show how the optimal extension period for the debt can be determined and illustrate that the debtholders' gain from extending the maturity date can be substantial. We present a valuation expression for the equity and show that the stockholders' wealth is not necessarily a monotone increasing function of the firm's value. This has the interesting implications that stockholders in struggling or highly levered firms may have an incentive to undertake projects with negative NPVs in order to reduce firm value. The intuition for this is that the benefit to the stockholders from the temporary protection from debtholders' claims given by an extension can more than offset the decline in the firm's value.

Section I derives closed-form expressions for the values of calls and puts that can be extended by the optionholder. Section II presents a number of examples and applications of these options. Section III derives valuation formulas for simple writer-extendible options and presents some additional examples. Section IV examines the pricing of flexible writer-extendible options, models the optimal extension problem, and discusses the pricing of equity in risky levered firms in the presence of bankruptcy costs. Section V summarizes the results and presents concluding remarks.

I. Pricing Extendible Options

In this section, we derive closed-form expressions for the values of extendible European options. These are options that can be exercised at their maturity date T_1 but that also allow the optionholder at time T_1 to extend the life of the option until T_2 by paying an additional premium of A to the option writer.[2] In deriving these expressions, we also allow the strike price of the option to be adjusted from K_1 to K_2 at the time of the extension.

A. The Valuation Framework

In order to apply standard valuation theory to these contingent claims, we make the following assumptions:

(A1) Markets are perfect in the sense that there are no transaction costs, restrictions on short sales, etc. Trading takes place continuously in time.

(A2) The underlying asset price X is governed by the following stochastic differential equation:

$$\frac{dX}{X} = \alpha dt + \sigma dZ, \tag{1}$$

where α and σ are constants and Z is a standard Wiener process. These dynamics imply that the underlying asset does not pay or receive any dividends or other types of cash flows.

(A3) The instantaneous riskless rate r is constant.

Given these assumptions, a simple hedging argument can be used to show that the price of any contingent claim $V(X, t)$ with payoffs that are functions of X and t satisfies the following valuation equation:

$$\frac{\sigma^2 X^2}{2} V_{XX} + rXV_X - rV + V_t = 0, \tag{2}$$

subject to the appropriate boundary and initial conditions. The prices for specific contingent claims such as extendible puts and calls can be obtained by first specifying the maturity conditions for these options and then solving the partial differential equation in (2).

B. Extendible Calls

We designate the current value of an extendible call by $EC(X, K_1, T_1, K_2, T_2, A)$. In addition, we denote the current value of an ordinary European call with strike price K and maturity T as $C(X, K, T)$. Using this notation, the maturity condition satisfied by the extendible call at T_1 is

$$\max(0, C(X, K_2, T_2 - T_1) - A, X - K_1). \tag{3}$$

[2] For notational simplicity, we assume that claims are valued as of time zero unless otherwise noted. This assumption results in no loss of generality since the dynamics for the underlying asset price are time homogeneous.

This payoff function allows the optionholder to choose the maximum of three different payoffs, instead of just two as in the case of a conventional call. Alternatively, (3) can be written as

$$\max(\max(0, C(X, K_2, T_2 - T_1) - A), \max(0, X - K_1)). \tag{4}$$

Expressing the payoff function in this form shows that the payoff function for an extendible call is the maximum of two risky payoffs: the payoffs for a conventional option and a call on a call (a compound option—see Geske (1979a)). In this respect, the payoff function for an extendible call is similar to that for an option on the maximum of two risky assets (Stulz (1982)). Note, however, that extendible calls and options on the maximum on two risky assets are fundamentally different securities and that their pricing formulas are not nested.[3] On the other hand, we will show later that the value of a compound option can be obtained as a special case of the extendible call option pricing formula.

The payoff function for an extendible call is illustrated in Figure 1. When $A > 0$, there is some critical value of X at time T_1, designated I_1, below which the option is not extended. In addition, there is another critical value, designated I_2, above which the option is again not extended. Thus, the option is extended if and only if X is in the interval $[I_1, I_2]$. If $X < I_1$ at T_1, the option expires out-of-the-money. If $X > I_2$ at T_1, the option is exercised rather than extended.

The exact values of I_1 and I_2 depend on the particular characteristics of the options involved. Although I_1 and I_2 can be expressed analytically in series form,[4] it is generally more convenient to determine their values directly from the maturity condition. For example, the value of I_1 is obtained by solving the following equation:

$$C(I_1, K_2, T_2 - T_1) = A. \tag{5}$$

It is easily shown that $A \leq I_1 \leq A + K_2 e^{-r(T_2 - T_1)}$. In the special case where $A = 0$, $I_1 = 0$ also. When $I_1 \geq K_1$, the option is never extended, the value of the extension privilege is zero, and the valuation problem is trivial. Consequently, we focus on the more interesting case where $I_1 < K_1$. A sufficient condition for $I_1 < K_1$ is $A < K_2 - K_2 e^{-r(T_2 - T_1)}$. A necessary condition for $I_1 < K_1$ is $A < K_1$. Implicit differentiation shows that I_1 is an increasing function of A and K_2 and a decreasing function of T_2, r, and σ^2. In a similar way, the value of I_2 is found by solving the following equation:

$$C(I_1, K_2, T_2 - T_1) = I_2 - K_1 + A. \tag{6}$$

When $I_1 < K_1$, (6) implies that $K_1 < I_2 \leq \infty$. If $A < K_1 - K_2 e^{-r(T_2 - T_1)}$, then $I_2 = \infty$. Differentiating (6) shows that I_2 is an increasing function of K_1, T_2, r, and σ^2 and a decreasing function of A and K_2.

[3] Stulz (1982) derives prices for options on the minimum or maximum of two risky assets. Although the payoff function in (4) involves two different risky payoffs, the extendible call cannot be priced using Stulz's results because the call price appearing in (4) does not follow a stationary Markov process—the underlying asset and call prices do not follow a bivariate geometric Brownian motion.

[4] This can be done by expanding option prices in a Taylor series expansion or by using the series expansion for the standard normal distribution function given in Abramowitz and Stegun (1972), Chapter 26.

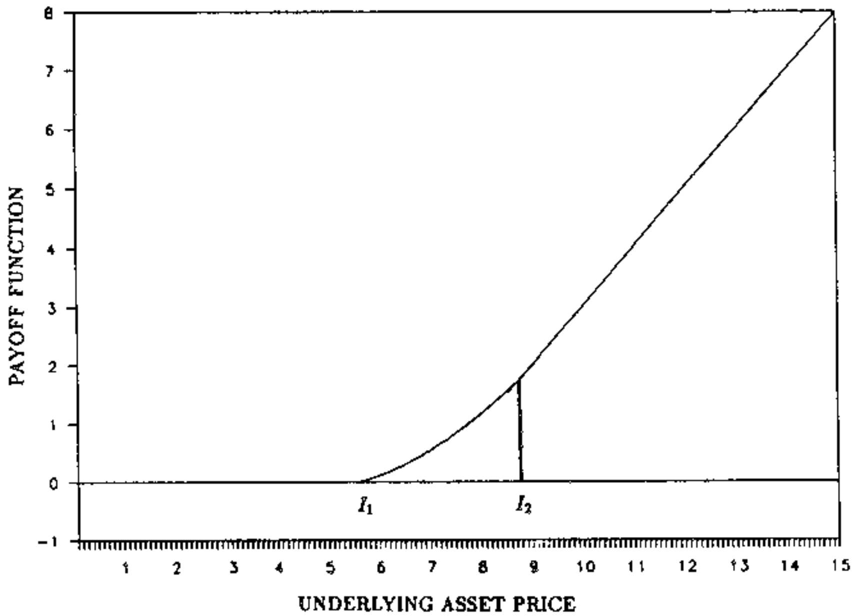

Figure 1. An example of the payoff function for an extendible call. I_1 is the critical value below which the call is not extended; I_2 is the critical value above which the call is exercised rather than extended.

Solving (2) subject to the maturity condition (3) gives the following closed-form expression for the extendible call:

$$\begin{aligned} &EC(X, K_1, T_1, K_2, T_2, A) \\ &\quad = C(X, K_1, T_1) + XN(\gamma_1, \gamma_2, -\infty, \gamma_3, \rho) \\ &\qquad - K_2 e^{-rT_2} N(\gamma_1 - \sqrt{\sigma^2 T_1}, \gamma_2 - \sqrt{\sigma^2 T_1}, -\infty, \gamma_3 - \sqrt{\sigma^2 T_2}, \rho) \\ &\qquad - XN(\gamma_1, \gamma_4) + K_1 e^{-rT_1} N(\gamma_1 - \sqrt{\sigma^2 T_1}, \gamma_4 - \sqrt{\sigma^2 T_1}) \\ &\qquad\qquad - Ae^{-rT_1} N(\gamma_1 - \sqrt{\sigma^2 T_1}, \gamma_2 - \sqrt{\sigma^2 T_1}), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \gamma_1 &= (\ln(X/I_2) + (r + \sigma^2/2)T_1)/\sqrt{\sigma^2 T_1}, \\ \gamma_2 &= (\ln(X/I_1) + (r + \sigma^2/2)T_1)/\sqrt{\sigma^2 T_1}, \\ \gamma_3 &= (\ln(X/K_2) + (r + \sigma^2/2)T_2)/\sqrt{\sigma^2 T_2}, \\ \gamma_4 &= (\ln(X/K_1) + (r + \sigma^2/2)T_1)/\sqrt{\sigma^2 T_1}, \\ \rho &= \sqrt{T_1/T_2}, \end{aligned}$$

$N(a, b, c, d, \rho)$ is the cumulative probability of the standard bivariate normal density with correlation coefficient ρ for the rectangular region $[a, b] \times [c, d]$, and $N(a, b)$ is the cumulative probability of the standard normal density in the interval $[a, b]$.[5] The first term in this expression is the value of a conventional call with strike price K_1 and maturity T_1. The sum of the remaining terms represents the value of the extension privilege.

A number of rational bounds can be placed on the value of the extendible call. For example, since the value of the extension privilege is nonnegative, the extendible call is worth at least as much as the corresponding nonextendible option. This lower boundary can be improved upon by observing that the maximum of the two payoff functions in (4) is greater than or equal to either of the two risky payoffs. Thus, the value of the extendible call is greater than or equal to the maximum of a conventional call with strike K_1 and maturity T_1 and a compound option on $C(X, K_2, T_2 - T_1)$ with strike price A. The maximum value of the extendible call is easily shown to be X.

The extendible call has a number of interesting special cases. For example, if $I_1 = 0$ and $I_2 = \infty$, then the call is always extended at T_1 and its value is $C(X, K_2, T_2)$. If $I_1 > 0$ and $I_2 = \infty$, then the extendible call reduces to a compound option on $C(X, K_2, T_2 - T_1)$ with strike price A. As $I_1 \rightarrow K_1$, the value of the extension privilege approaches zero and the value of the extendible call is just $C(X, K_1, T_1)$. The same is also true if $A = 0$ and $K_2 \rightarrow \infty$. Of course, the value of the extendible call is zero if $X = 0$.

In deriving comparative statics, we focus first on the value of the extension privilege. (Note that all of the comparative statics results reflect the fact that I_1 and I_2 vary as the underlying variable changes.) The extension privilege is easily shown to be an increasing function of K_1 and T_2 and a decreasing function of A and K_2. An increase in the X, however, has an indeterminate effect on the value of the extension privilege. This is shown in Figure 2, which plots the value of the extension privilege as a function of the underlying asset price for various combinations of parameters. As illustrated, the extension privilege is worth relatively little for deep-out-of-the-money options, increases in value as X approaches the optimal extension range $[I_1, I_2]$, and then decreases for larger values of X. The intuition for this is that the call is more likely to be extended when the current value of X is near the optimal extension range. The comparative statics for T_1, r, and σ^2 are also indeterminate. This might seem counterintuitive at first because increases in r and σ^2 and decreases in T_1 not only increase the length of the optimal extension interval $[I_1, I_2]$ but also increase the payoffs from an extension at every point in this interval. The reason for these indeterminate comparative statics is that the changes in T_1, r, and σ^2 also affect the risk-neutral conditional density of S at T_1 and, therefore, the probability that an extension will occur. Thus, even though a change in T_1, r, and σ^2 can increase the payoff from an extension, the probability that an extension occurs could decline sufficiently to more than offset the increased payoff and lower the overall value of the extension privilege.

The comparative statics for the extendible call are generally similar to those for nonextendible calls. For example, the extendible call is an increasing function

[5] The probability $N(a, b, c, d, \rho)$ can be determined directly from the standard bivariate normal distribution function $N(\cdot, \cdot, \rho)$ by the relation $N(a, b, c, d, \rho) = N(b, d, \rho) - N(a, d, \rho) - N(b, c, \rho) + N(a, c, \rho)$. Similarly, the probability $N(a, b)$ is equal to $N(b) - N(a)$, where $N(\cdot)$ is the standard normal distribution function.

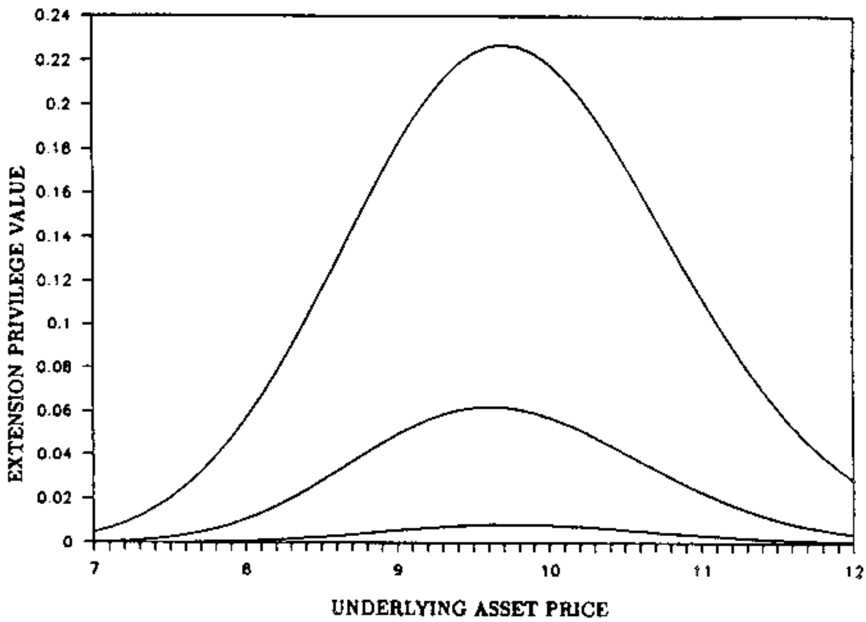

Figure 2. Value of the extension privilege as a function of the underlying asset price. In each case, the risk-free rate is .06 per annum, the variance of the underlying asset's returns is .04 per annum, the initial maturity of the option is .2 year, the extended maturity is 1 year, and the original strike price is 10. From bottom to top, the extension fee and the extended strike price are .05 and 13, .04 and 12, and .03 and 11, respectively.

of X, T_1, r, and σ^2 and a decreasing function of K_1. However, the sensitivity of the extendible call value to each of these variables can be quite different. This is illustrated in Figure 3, which plots extendible and nonextendible call values against the underlying asset price. As shown, the extendible call is more sensitive to changes in X than a conventional call for out-of-the-money options but is less sensitive for in-the-money options—the deltas for extendible calls can differ substantially from those for nonextendible calls. Finally, the extendible call is also an increasing function of T_2 and a decreasing function of A and K_2.

C. Extendible Puts

The analysis for extendible puts is very similar to that for extendible calls. We denote the value of an extendible put by $EP(X, K_1, T_1, K_2, T_2, A)$. Similarly, the value of an ordinary European put with strike price K and maturity T is $P(X, K, T)$. The maturity condition satisfied by an extendible put at T_1 is

$$\max(0, P(X, K_2, T_2 - T_1) - A, K_1 - X). \tag{8}$$

This payoff function can again be written as the maximum of two risky payoffs:

$$\max(\max(0, P(X, K_2, T_2 - T_1) - A), \max(0, K_1 - X)). \tag{9}$$

Thus, at T_1, the extendible put allows the optionholder to choose between the payoff function for a call option on a put and the payoff function for an ordinary put option.

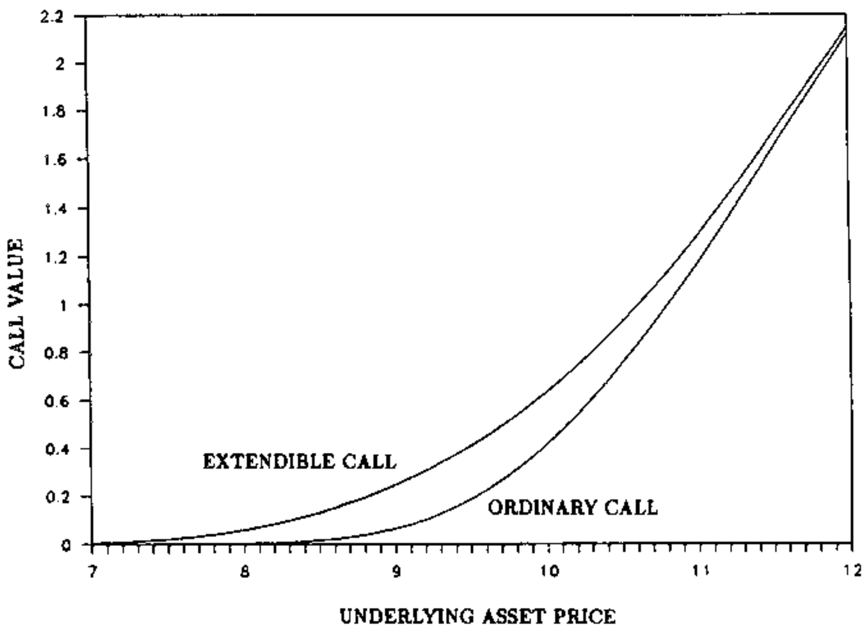

Figure 3. Values of an extendible call and an ordinary call as a function of the underlying asset's price. The risk-free rate is .06 per annum, the variance of the underlying asset's return is .04 per annum, the initial maturity is .2 year, and the initial strike price is 10. The extended maturity date is 1 year, the extended strike price is 11, and the extension fee is .03.

As for extendible calls, the payoff function for the extendible put implies that the put is extended at T_1 if and only if the underlying asset price is in the range $[I_1, I_2]$. When $X < I_1$, the put is exercised at T_1. When $X > I_2$, the put is allowed to expire at time T_1. The optimal extension parameters I_1 and I_2 can be found by solving the following two equations:

$$P(I_1, K_2, T_2 - T_1) = K_1 - I_1 + A, \tag{10}$$

$$P(I_2, K_2, T_2 - T_1) = A. \tag{11}$$

Although we use the notation I_1 and I_2 for both extendible calls and puts, it is important to note that the optimal extension regions for calls and puts generally do not coincide even if the other parameters are the same. This fact precludes us from deriving a simple put-call parity relation for extendible options. An analysis of the payoff function in (8) shows that $0 \le I_1 \le K_1$. In addition, I_1 is an increasing function of A, K_1, and r and a decreasing function of K_2, T_2, and σ^2. When $I_2 < K_1$, the put is never extended, the value of the extension privilege is zero, and the valuation problem is again trivial. Accordingly, we focus on the case where $K_1 \le I_2 \le \infty$. Note that $I_2 = \infty$ if and only if $A = 0$. I_2 is an increasing function of K_2, T_2, and σ^2 and a decreasing function of A and r.

Solving (2) subject to the maturity condition (8) gives the following expression for the value of the extendible put:

$$
\begin{aligned}
&EP(X, K_1, T_1, K_2, T_2, A) \\
&\quad = P(X, K_1, T_1) - XN(\gamma_1, \gamma_2, -\infty, \gamma_3, \rho) \\
&\qquad + K_2 e^{-r(T_2-T_1)} N(\gamma_1 - \sqrt{\sigma^2 T_1}, \gamma_2 - \sqrt{\sigma^2 T_1}, -\infty, \gamma_3 - \sqrt{\sigma^2 T_2}, \rho) \\
&\qquad + XN(\gamma_4, \gamma_2) - K_1 e^{-rT_1} N(\gamma_4 - \sqrt{\sigma^2 T_1}, \gamma_2 - \sqrt{\sigma^2 T_1}) \\
&\qquad\qquad - Ae^{-rT_1} N(\gamma_1 - \sqrt{\sigma^2 T_1}, \gamma_2 - \sqrt{\sigma^2 T_1}), \qquad (12)
\end{aligned}
$$

where the γ terms and ρ are defined in (7). The first term in this expression is again the value of an ordinary option without the extension feature, and the sum of the remaining terms represents the value of the extension privilege. Following the previous analysis, the lower bound for the value of the extendible put is the maximum of the nonextendible put value $P(X, K_1, T_1)$ and the value of a call with strike price A and maturity T_1 on the put value $P(X, K_2, T_2 - T_1)$. The upper bound for the extendible put is $\max(K_1 e^{-rT_1}, K_2 e^{-rT_2} - Ae^{-rT_1})$.

The extendible put also has a number of special cases. For example, if $I_1 = K_1$, the put is never extended and the value of the extendible put is simply $P(X, K_1, T_1)$. If both A and I_1 equal zero, the put is always extended and has value $P(X, K_2, T_2)$. If $A > 0$ and $I_1 = 0$, the value of the extendible put is equal to the value of a call on $P(X, K_2, T_2 - T_1)$ with strike price A. Turning to the comparative statics for the extension privilege, it is readily shown that this value is an increasing function of K_2 and T_2 and a decreasing function of A and K_1. The partial derivatives with respect to X, T_1, r, and σ^2 are again indeterminate for the reasons discussed earlier for extendible calls. The value of the extendible put is an increasing function of K_1, K_2, T_1, T_2, and σ^2 and a decreasing function of X, A, and r.

II. Applications of Extendible Options

There are many examples of financial contracts that explicitly incorporate extendible options. In addition, financial contracts that allow various degrees of flexibility over the timing of cash flows are growing rapidly in importance and popularity—these types of contracts often contain implicit extendible options. This section discusses several examples where the optionholder has the right to extend the life of the contract. Examples where the option writer may extend the life of the contract (writer-extendible options) are described in a later section.

A. *Real Estate Options*

Options to purchase or sell real estate frequently allow the optionholder the right to extend the life of the option by paying an additional fixed fee to the option writer at the time of the extension. This right is particularly important in land development when the value of a parcel of property X to a developer may well depend on the likelihood of obtaining adjacent properties to the optioned property or the possibility of obtaining a zoning change. If these uncertainties are not completely resolved at the initial maturity date T_1, X may be in the optimal extension range, and the developer would extend the life of the option until T_2 by paying an additional amount of A to the property owner. The option often provides for a specific increase in the purchase price of the property (the strike price) at the time of the option extension. These types of options can be valued directly using the closed-form expression derived in the previous section.

An interesting variant of an extendible option has recently become popular in real estate acquisition for tax reasons. In form, this type of option is actually a chain of sequential conventional options where the prices of future options are determined at the beginning of the life of the first option. A typical example might be a contract that grants a prospective purchaser a one-year option with the provision that a second one-year option could be obtained at the end of the first year by paying an additional amount of A. Although two separate options are involved, this contract can be valued simply as an extendible option with $T_1 = 1$ and $T_2 = 2$.

B. Extendible Bonds

Black and Scholes (1973), Merton (1974), Smith (1979), and others have shown that the stockholders' claim on the assets of a levered firm is a call option on the value of the firm, where the maturity of the call is determined by the maturity date of the debt. Recently, however, many firms have issued bonds that allow the firm the right to extend the maturity date of the bonds. These extendible bonds have become especially popular as financing tools for leveraged buyouts (see *Corporate Finance,* April 1989, p. 39). Observe that, in extending the life of the firm's maturing bonds, the stockholders extend the maturity of their call on the value of the firm. Thus, the stockholders effectively have an extendible call on the firm's value. This extension right could be particularly important in the event that the firm's value is less than the face amount of the debt at the initial maturity date (a very real possibility for many junk bonds)—the extension privilege allows the stockholders to "buy" additional time to turn the firm around rather than losing control to the bondholders. Note that the same line of reasoning could be used to show that many types of corporate reorganizations—such as Chapter 11 bankruptcy—can be viewed as the exercise of an implicit extension privilege (for example, see Franks and Torous (1989)). This suggests that the extendible option analysis has many potential applications in pricing the capital structure of a firm.

C. Warrants

Warrants have long played an important role in the capital structure of many firms. For example, warrants are routinely issued by corporations to lenders, investment bankers, and executives in order to increase the compatibility of their incentives with those of the stockholders. In addition, warrants are often an important financing tool (see Smith (1977) and Lauterbach and Schultz (1989)).

Recently, a number of longer-term warrants issued by firms have provisions that adjust the strike price of the warrant periodically. For example, the strike price of the OTC-traded Action Products International warrants expiring December 1, 1991 changes from 2.75 to 3.00 on December 1, 1990. These types of warrants can be modeled as extendible calls. To see this, denote the date of the strike price change from K_1 to K_2 as T_1, and denote the maturity date of the warrant as T_2. At T_1, the optionholder must decide whether to exercise the warrant early at the strike price K_1. Note that the early exercise of the warrant can be optimal when K_2 is greater than K_1 even if the underlying asset does not pay dividends.[6] If the investor does not exercise the warrant at T_1, the warrant is automatically ($A = 0$) extended until T_2. The pricing formula for extendible calls can be used to value these types of warrants.[7]

D. American Options with Stochastic Dividends

As a further application of extendible options, consider an American call with expiration date T_2 on a stock which pays a single dividend at time T_1. Roll (1977), Geske (1979b), and Whaley (1981) derive a closed-form expression for the value of the American call in the case where the size of the dividend paid is known. However, since firm values are stochastic, a more realistic model of American call values would also allow the dividend to be stochastic.

In deriving closed-form expression for American call values, we make the economically realistic assumption that the firm has a target dividend payout ratio of α. Thus, at time T_1, the firm pays a dividend of αX per share.[8] Since the actual value of X is unknown when the American call is purchased, the size of the dividend paid is a random variable. The payoff function for the holder of an option at T_1 can be expressed as

$$\max(0,\ C((1-\alpha)X,\ K,\ T_2 - T_1),\ X - K), \tag{13}$$

where X is the cum-dividend stock value and K is the strike price. Using the well-known first-order homogeneity property of call prices,[9] this can be written as

$$\max(0,\ (1-\alpha)C(X,\ K/(1-\alpha),\ T_2 - T_1),\ X - K), \tag{14}$$

which is the payoff function for an extendible call with $A = 0$ and with the provision that only $(1-\alpha)$ calls are extended. In this context, $K_1 = K$, $K_2 = K/(1$

[6] Intuitively, increasing the exercise price has an effect on the value of the option that is similar to the payment of a dividend—an increase in the exercise price lowers the value of the option. By exercising early, the warrantholder avoids the decline in the value of the option at the cost of foregoing the time premium for the option—early exercise is optimal if the decline exceeds the foregone time premium. We abstract from the strategic warrant exercise issues raised by Emanuel (1983) and Constantinides (1984) and assume that warrantholders are competitive.

[7] If the dilution resulting from the exercise of the warrant is negligible, the value of the warrant is given directly by (7). If not, the extendible call valuation expression is easily modified to correct for the dilution effects and the increase in firm value resulting from the premium payment, using a technique similar to that described in Section II.D for American options.

[8] For simplicity, we assume that the stock value declines by the amount of the dividend on the ex-dividend date. However, ex-dividend date stock price adjustments of D percent of the dividend amount are easily handled by replacing α with αD in the analysis.

[9] This homogeneity property is discussed by Merton (1973).

$-\alpha$), and the valuation formula can be obtained directly from (7) by multiplying the second and third terms (the terms involving the cumulative bivariate normal density terms) by $(1-\alpha)$. Note that, since A is zero, I_1 is also zero. The value of I_2 is found by solving the expression:

$$(1-\alpha)C(I_2, K/(1-\alpha), T_2-T_1) = I_2 - K. \tag{15}$$

The value of I_2 is the critical value of X above which the American call is exercised early at T_1. It is easily shown that, when $\alpha > 0$, $I_2 < \infty$. Thus, there is always a nonzero probability of early exercise.[10] The early exercise bound I_2 is an increasing function of r, σ^2, and T_1 and a decreasing function of α and T_2.

E. Shared-Equity Mortgages

A standard financing vehicle in commercial real estate lending is the shared-equity mortgage. A shared-equity mortgage is an ordinary mortgage with the additional feature that the lender shares in any appreciation in the property above and beyond the face value of the loan. Thus, a shared-equity loan can be viewed as a portfolio consisting of an ordinary mortgage and a call option on the value of the underlying property. In exchange for the call option, the lender usually requires a lower interest rate on the mortgage.

In its simplest form, a shared-equity mortgage requires the property owner to pay the face amount of the debt, along with the lender's share of the appreciation on the property, at the maturity date of the mortgage T_1. In order to do this, the property owner must generally refinance the loan or sell the property. However, a number of shared-equity mortgages on commercial properties have the provision that, if the lender chooses to refinance the loan, the lender's call option on the value of the property is also extended. Thus, the lender's original call option on the property is effectively an extendible call and can be valued accordingly.

III. Pricing Simple Writer-Extendible Options

So far, we have focused on the valuation of options when the optionholder has the right to extend the expiration date. However, many financial contracts either implicitly or explicitly incorporate options that can be extended by the option writer. For example, corporate warrants often give the issuing firm the right to extend the life of the warrants if out-of-the-money at the initial expiration date. This right can be valuable to the option writer. For example, if the writer faces a substantial tax penalty when the option expires out-of-the-money, the writer may have a strong incentive to extend the life of an out-of-the-money option in the hope that the option will subsequently expire slightly-in-the-money. In this section, we derive closed-form expressions for the values of simple writer-extendible call and put options. These are options that can be exercised at their initial maturity date T_1, but are extended to T_2 if out-of-the-money at T_1. In addition, we present several specific examples of simple writer-extendible options and discuss the writer's economic incentives for extending option maturities.

[10] See Samuelson (1965) and Merton (1973).

A. *Writer-Extendible Calls*

We designate the currrent value of a simple writer-extendible call as $WC(X, K_1, T_1, K_2, T_1)$, where T_1 is the initial maturity date, T_2 is the extended maturity date, and the strike price is adjusted from K_1 to K_2 if the call is extended. Because calls are extended by the writer when $X < K_1$ at T_1 in this framework, we assume that no additional amounts are paid by the optionholder in the event of an extension. Using this notation, the boundary condition satisfied by the writer-extendible call at T_1 is

$$WC(X, K_1, T_1, K_2, T_2) = \begin{cases} C(X, K_2, T_2 - T_1), & \text{if } X < K_1 \text{ at } T_1, \\ X - K_1, & \text{if } X \geq K_1 \text{ at } T_1. \end{cases} \quad (16)$$

Intuitively, this payoff function means that, if the option is out-of-the-money at the initial expiration date, the optionholder receives a second chance to exercise the option at T_2. This payoff function is fundamentally different from the payoff function for the extendible call given in (3). For example, (16) is discontinuous at $X = K_1$, while the extendible call payoff function is continuous at every point. This discontinuity plays an important role in determining the analytical properties of writer-extendible calls. Despite the differences in the properties of the payoff functions, however, the valuation formula for the writer-extendible call is given directly from (7) by imposing the restrictions $A = 0$, $I_1 = 0$, and $I_2 = K_1$:

$$\begin{aligned} WC(X, K_1, T_1, K_2, T_2) &= C(X, K_1, T_1) + XN(\gamma_3, -\gamma_4, -\rho) \\ &\quad - K_2 e^{-rT_2} N(\gamma_3 - \sqrt{\sigma^2 T_2}, -\gamma_4 + \sqrt{\sigma^2 T_1}, -\rho), \end{aligned} \quad (17)$$

where $N(\cdot, \cdot, -\rho)$ is the standard bivariate normal distribution function with correlation $-\rho$.

The writer-extendible call has a number of interesting properties. For example, unlike ordinary call options or even the extendible call value derived in (7), the writer-extendible call is not always a monotone increasing function of the underlying asset price. The intuitiion for this surprising result is that, if the writer-extendible call is near-the-money as it approaches its initial maturity date, the optionholder would rather have a second chance to exercise the option than the small payoff associated with a slightly-in-the-money expiration. This follows from the discontinuity of the payoff function.[11] Some examples of the relation between writer-extendible call prices and the underlying asset price are shown in Figure 4. As illustrated, the writer-extendible call value can have both a hump and a trough when graphed as a function of the underlying asset price. An important implication of this property is that financial contracts that incorporate these types of options can result in very different incentive structures from financial contracts that include conventional options.[12] Note also from Figure 4

[11] The left limit of the payoff function for the writer-extendible call as $X \rightarrow K_1$ is strictly larger than the corresponding right limit if $T_2 > T_1$. Thus, because of the convergence (in L^2) of the extendible call to its payoff function at $t \rightarrow T_1$, the extendible call must be a decreasing function of x for some maturities.

[12] For example, if executives are given extendible calls or warrants as part of their compensation, their incentives are not always consistent with the shareholders' interests. Lauterbach and Schultz (1989) examine a large sample of daily warrant prices during the 1971–1980 period and find evidence that warrant prices are not uniformly monotone increasing functions of the underlying stockprice.

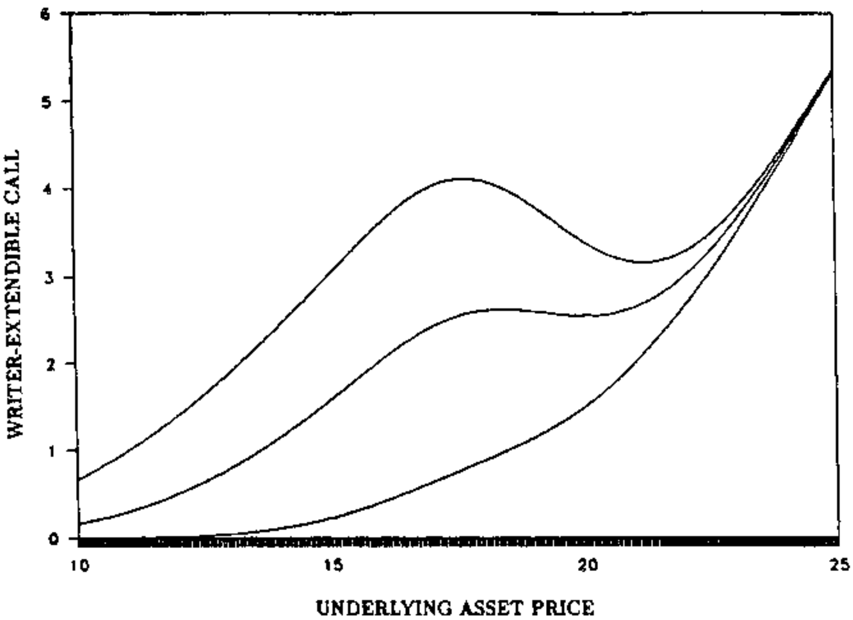

Figure 4. Writer-extendible call prices as a function of the underlying asset price. In each case, the risk-free rate is .07 per annum, the variance of the underlying asset's returns is .04 per annum, the initial maturity date is .25 years, and the strike price is 20 (the strike price is not adjusted if the call is extended). From bottom to top, the extended maturities for the calls are 1, 3, and 5 years, respectively.

that the writer-extendible call price can be both convex and concave in the underlying asset price.

Since changes in K_1 and T_1 also affect the likelihood that the optionholder has a second chance to exercise the option, the intuition for why the signs of the partial derivatives of the writer-extendible call with respect to these parameters are indeterminate is similar to that described above. On the other hand, since changes in K_2 and T_2 do not affect the probability of an extension, the partial derivatives of the writer-extendible call with respect to K_2 and T_2 can be signed and are less than zero and greater than zero, respectively. Finally, changes in r and σ^2 can have very complex effects on writer-extendible call values because changes in these parameters not only influence the likelihood of the optionholder having a second chance to exercise the option but also affect the drift of the risk-neutral process. The partial derivative of the writer-extendible call with respect to the risk-free rate can change signs as many as two times. In contrast, conventional European calls are monotonically increasing functions of the risk-free rate. In addition, a writer-extendible call can actually be a decreasing function of σ^2 in some situations. This property is strikingly different from the relation between volatility and ordinary call option prices.

As in Merton (1973), a number of rational bounds can be placed on the value of a writer-extendible call. For example, since the value of a writer-extendible call is equal to the value of an ordinary call plus the value of the second chance to exercise the option, the writer-extendible call cannot be worth less than the value of an ordinary call with strike price K_1 expiring at the initial expiration date T_1. Since a lower bound for the value of an ordinary call is $\max(0, X - K_1 e^{-rT_1})$, this is a lower bound for the value of the writer-extendible call as well. As with ordinary calls, the value of the writer-extendible call cannot exceed the value of the underlying asset. This implies that, since the value of an ordinary call approaches X as $T_1 \to \infty$, $K_1 \to 0$, or $\sigma^2 \to \infty$, the value of the extension feature must approach zero as $T_1 \to \infty$, $K_1 \to 0$, or $\sigma^2 \to \infty$.

B. Writer-Extendible Puts

We designate the current value of a writer-extendible put by $WP(X, K_1, T_1, K_2, T_2)$. The payoff function for the writer-extendible put is

$$WP(X, K_1, T_1, K_2, T_2) = \begin{cases} K_1 - X, & \text{if } X < K_1 \text{ at } T_1, \\ P(X, K_2, T_2 - T_1), & \text{if } X \geq K_1 \text{ at } T_1. \end{cases} \tag{18}$$

As before, the payoff function is a discontinuous function of the underlying asset price and indicates that the value of a writer-extendible put can be decomposed into two components: an ordinary put option and the value of the extension feature. Proceeding as before, the value of the writer-extendible put is given by substituting the parameter values $A = 0$, $I_1 = K_1$, and $I_2 = \infty$ into (12):

$$\begin{aligned} WP(X, K_1, T_1, K_2, T_2) &= P(X, K_2, T_2 - T_1) \\ &\quad + K_2 e^{-rT_2} N(-\gamma_3 + \sqrt{\sigma^2 T_2}, \gamma_4 - \sqrt{\sigma^2 T_1}, -\rho) - XN(-\gamma_3, \gamma_4, -\rho). \end{aligned} \tag{19}$$

Again, most of the comparative statics for the writer-extendible put are indeterminate. For example, the value of a writer-extendible put can be an increasing function of the underlying asset price over some ranges. Similarly, the partial derivatives of the extendible put with respect to K_1, T_2, r, and σ^2 can be positive or negative.

C. Examples of Simple Writer-Extendible Options

Corporate warrants provide an intriguing example of a writer-extendible option. Recall that, by issuing warrants, a corporation is essentially writing call options on the stock of the firm, and, as mentioned earlier, warrants often give the issuing firm the right to unilaterally extend the maturity date. This right is frequently exercised by corporations—the Commerce Clearing House *Capital Changes Reporter* lists over 500 cases where corporations extended the life of expiring warrants during the 1975–1988 period.[13]

At first, it may appear paradoxical that a corporation would choose to extend the life of an expiring out-of-the-money warrant since this seems to be a pure wealth transfer from the current stockholders to the warrantholder. However, a firm could rationally choose to extend the maturity of its expiring warrants if the firm faced some penalty (not part of the actual warrant contract) associated

[13] This estimate is based on a random sample drawn from the 14,700 pages of the *Capital Changes Reporter*. By actual count, there were 135 warrant extensions reported during the 1987–1988 period alone.

with an out-of-the-money expiration. As a specific example of this type of situation, observe that, for warrants which expired out-of-the-money prior to July 18, 1984, the premium received by the firm when the warrants were originally issued was taxable as income.[14] On the other hand, if the warrants were exercised, they were considered to be part of the capital transaction and the initial premium was not taxable. Thus, there was a potentially large tax penalty to the firm if the warrants expired out-of-the-money. Faced with the prospect of this penalty, the firm would have a strong incentive to extend the life of the expiring warrants in the hope that the warrants would later expire in-the-money. Another situation where a firm with outstanding warrants might face a penalty for an out-of-the-money expiration would be the case where the firm is committed to issuing stock. If the warrants are allowed to expire out-of-the-money, the firm faces the substantial transaction costs[15] associated with an equity offering. However, if the warrants are extended and subsequently expire in-the-money, then the new shares can be issued at a price close to the market price with little or no marginal cost. In this situation, extending the warrants is similar to the strategy[16] of calling convertible bonds in order to force conversion—while extending the maturities of warrants does not guarantee that the warrants will subsequently be exercised, it does provide the warrantholder a second chance to do so.

An interesting variation of a simple writer-extendible option is given by the frequently-used tax-planning device of a lease with an option to purchase. For example, assume that the owner of a substantially appreciated asset wishes to sell. By selling the asset outright, the owner faces large capital gains taxes because the tax basis of the property is much less than its fair market value. However, if the actual transfer of title can be postponed until after the owner's death, the tax basis is increased to the fair market value and the estate's income tax on the sale is substantially less.[17] One way to achieve this deferral is for the owner to lease the property with the option to purchase, but with the provision that the option writer can extend the maturity of the option if it is in-the-money at T_1. If the owner is still alive at T_1, the option may be extended—if not, the option can be exercised. This type of a writer-extendible call can be valued by substituting in the values $A = 0$, $I_1 = K_1$, and $I_2 = \infty$ in (7).

IV. Pricing Flexible Writer-Extendible Options

In the previous section, we derived expressions for simple writer-extendible calls and puts—options that are extended for a given period if out-of-the-money at the initial expiration date. However, in some types of financial contracts, the

[14] Prior to 1976, the initial premium received by the firm for the warrants was recognized as ordinary income when the warrants expired unexercised. Subsequently, the initial premium was treated as a short-term capital gain by the firm when the warrants elapsed. See Turov (1974).

[15] Smith (1977) documents that the costs associated with an underwritten equity offering can be as large as 15% of the proceeds for small firms and averages 6% for all firms in a sample of 484 offerings during 1971–1975.

[16] See Ingersoll (1976, 1977) and Constantinides and Grundy (1986) for discussions of this strategy.

[17] There may, of course, be other factors such as estate taxes that affect the profitability of this strategy.

writer also has the flexibility to choose the length of the extension period[18] at T_1. In this situation, the option writer selects the extension period that minimizes the net gain from an extension. If the optimal extension period is zero, then the option is not extended. Otherwise, the option is extended for the optimal period. As a result, T_2 is endogenous. We designate these types of contingent claims as flexible writer-extendible options. In this section, we show how the optimal extension period for a flexible writer-extendible option can be determined and illustrate how these claims can be priced.

To make the discussion more intuitive, we focus on a specific example of a flexible writer-extendible option—an equity position in a risky levered firm where bondholders have an incentive to extend the maturity date of the debt. Consider a firm with a simple capital structure consisting of equity and a single issue of discount bonds with maturity T and face value F. Denote the value of the firm X. In the absence of liquidation or bankruptcy costs, the payoff function for the stock is simply $\max(0, X - F)$, and the stock can be viewed as a call option on the firm. Consequently, the debt can be viewed as a long position in the firm and a short position in a call; the payoff function for the debt is $\min(X, F)$.

This analysis is altered if there are bankruptcy costs—that is, if the value of the firm as a going concern is greater than its liquidation value. Denote the percentage realization of the firm's assets in a liquidation situation by the parameter β, where $0 < \beta < 1$. Thus, if $X < F$ at the maturity date of the bonds, the bondholders receive only βX if they take over the firm. On the other hand, if they choose to extend the maturity of the defaulting bonds for an additional τ periods (until $T + \tau$), the bondholders in effect swap a known payoff of βX at T for a contingent claim that pays βX at $T + \tau$ if $X < F$, and F otherwise.[19] If the value of this contingent claim is greater than βX, the bondholders extend the maturity of the debt. We designate the difference between the value of this contingent claim and βX as the extension gain function $H(X, F, \tau)$ (which can be negative). Intuitively, the benefit of extending the maturity of the defaulting bonds is that the firm value may subsequently rise and allow the bondholders to avoid the liquidation costs of $(1 - \beta)X$. The cost of extending the maturity of the debt, of course, is the lost interest on the amount recovered.

Using this notation, it is clear that the defaulting bonds are extended if and only if

$$\max_{0 \leq \tau < \infty} H(X, F, \tau) > 0. \tag{20}$$

Thus, the bondholder first determines the value of τ that maximizes $H(X, F, \tau)$ and then extends the maturity of the bonds by τ if the maximized value of $H(V,$

[18] For example, there were a number of cases in the early 1970's where firms extended warrants without having the explicit contractual right to do so. Clearly, in these situations, the length of the extension period would be governed entirely by the corporation. See Noddings (1973).

[19] For reasons of tractability, we assume that the bondholders can extend the maturity date of the debt only once. However, this assumption is not economically unrealistic if an extension is viewed as a corporate reorganization such as a Chapter 11 bankruptcy or if there are additional fixed transaction costs to the lender resulting from an extension.

$F, \tau)$ is greater than zero.[20] A closed-form expression for $H(X, F, \tau)$ is obtained by solving (2) subject to the appropriate maturity condition and then subtracting βX:

$$H(X, F, \tau) = -\beta X N(\phi) + Fe^{-r\tau}N(\phi - \sqrt{\sigma^2\tau}), \tag{21}$$

where

$$\phi = (\ln(X/F) + (r + \sigma^2/2)\tau)/\sqrt{\sigma^2\tau}$$

and $N(\cdot)$ is the cumulative standard normal distribution function. The extension profit function equals 0 when $\tau = 0$ and equals $-\beta X$ for $\tau = \infty$. If $\beta = 1$, it is easily shown that the maturity of the debt is never extended. Alternatively, if $\beta < 1$, an application of l'Hôpital's rule shows that $H(X, F, \tau) > 0$ for some $\tau > 0$. Hence, if there are positive liquidation costs, the bondholders always prefer to extend the maturity of the defaulting bonds rather than instigate bankruptcy proceedings.

The optimal extension period τ that maximizes $H(X, F, \tau)$ is easily determined numerically.[21] Substituting the optimal τ into $H(X, F, \tau)$ gives the amount of the extension gain to the bondholder. Table I presents examples of the gain from an optimal extension for different combinations of X and β. As illustrated, the extension gain is a decreasing function of the value of the firm at the time of the default. The reason for this is that the probability that the firm value exceeds F at the extended maturity date is a decreasing function of the firm's value at T. Note that the extension gain can be a substantial proportion of the liquidation costs. For example, if $X = 38$, $F = 40$, and $\beta = .65$, the gain from an extension is over one-third of the liquidation costs—by extending the maturity of the debt, the bondholders reduce the cost of a default by over one-third. Table II presents the corresponding values of the optimal extension period. As shown, the optimal extension period is a monotone decreasing function of the value of the firm at the time of default. This is intuitive since a firm that is in deep default would need a longer time to recover than a firm that is just slightly in default.

Note that, if the firm is in default at T, both the bondholder and the stockholder are better off if the bondholder extends the maturity date optimally. Given the Modigliani-Miller proposition, how is this possible? The answer to this is simply that there are really three classes of claimants to the residual value of the firm in this setting: the bondholder, the stockholder, and the lawyers (assuming for simplicity that the liquidation costs are legal fees). Thus, by extending the maturity of the defaulting debt, the bondholder in effect benefits himself (or herself) and the stockholder by expropriating the lawyers' claim on the assets of the firm.

We now turn our attention to the issue of how to value the equity in a firm where the debt is extended optimally in the event of a default. Since the optimal

[20] Note that, because the contingent claim issued by the bondholders by extending the maturity of the defaulting bonds can be valued using a preference-free approach, the optimal decision does not depend on the preferences of the bondholders.

[21] Implicit differentiation shows that the optimal extension period is unique and yields a maximum for $H(X, F, \tau)$.

Table I

Net Gain[a] to Bondholder from Optimally Extending the Maturity of a Defaulting Corporate Discount Bond for Various Values of the Realization Percentage of Assets in a Liquidation and Values of the Firm at the Time of Default[b]

Realization	Firm Value at Time of Default									
Percentage	20	22	24	26	28	30	32	34	36	38
95	.000	.000	.000	.001	.002	.004	.010	.024	.058	.158
90	.005	.009	.015	.026	.043	.071	.117	.195	.333	.614
85	.035	.054	.080	.118	.171	.247	.357	.519	.773	1.223
80	.111	.156	.215	.293	.395	.530	.712	.963	1.330	1.932
75	.243	.323	.424	.549	.707	.907	1.165	1.505	1.979	2.715
70	.431	.554	.702	.881	1.099	1.367	1.701	2.129	2.703	3.560
65	.676	.846	1.046	1.282	1.563	1.900	2.309	2.822	3.491	4.456

[a] The net gain for extending the maturity of a defaulting discount bond is the difference between the value of the optimally extended bond and the amount that the bondholder would receive if a liquidation occurred (the realization percentage multiplied by the value of the firm at the time of default). The optimal extension period is the extension period that maximizes the net gain from an extension.

[b] These values assume that the face amount of the discount bond is 40, that the risk-free interest rate is 6 percent per annum, and that the standard deviation of returns on the firm is 20 percent per annum.

Table II

Optimal Extension Periods[a] for Defaulting Corporate Discount Bonds for Various Values of the Realization Percentage of Assets in a Liquidation and Values of the Firm at the Time of Default[b]

Realization	Firm Value at Time of Default									
Percentage	20	22	24	26	28	30	32	34	36	38
95	.80	.73	.61	.52	.43	.35	.27	.20	.13	.06
90	1.62	1.40	1.20	1.02	.84	.69	.54	.40	.26	.13
85	2.37	2.06	1.76	1.50	1.25	1.02	.80	.60	.40	.20
80	3.10	2.69	2.32	1.98	1.66	1.36	1.07	.80	.54	.28
75	3.81	3.32	2.87	2.45	2.06	1.70	1.35	1.02	.69	.37
70	4.51	3.94	3.41	2.93	2.47	2.04	1.63	1.24	.85	.47
65	5.20	4.55	3.95	3.40	2.88	2.39	1.92	1.47	1.02	.57

[a] The optimal extension period is the length of time to extend the maturity date of a defaulting corporate discount bond that maximizes the difference between the value of the extended discount bond and the amount that a bondholder would receive if a liquidation occurred (the realization percentage multiplied by the value of the firm at the time of default).

[b] These values assume that the face amount of the discount bond is 40, that the risk-free interest rate is 6 percent per annum, and that the standard deviation of returns on the firm is 20 percent per annum.

extension period is a function of X only (F, r, T, σ^2, and β are constants), it can be expressed as $\tau(X)$. The payoff function for the equity at time T can now be expressed as $C(X, F, \tau(V))$ if $X < F$ and as $X - F$ if $X \geq F$. Since no new state variables are introduced into the analysis, the current value of the equity satisfies (2). From Theorem 5.3 of Friedman (1975), the current value of the equity can be expressed as

$$C(X, F, T) + e^{-rT} \int_0^F C(X, F, \tau(X))P(X)\, dX, \tag{22}$$

where $P(X)$ is the risk-neutral density[22] of X at time T conditional on its current value (denoted X_0):

$$P(X) = \frac{1}{X\sqrt{2\pi\sigma^2 T}} \exp\left(\frac{-(\ln(X/X_0) - (r - \sigma^2/2)T)^2}{2\sigma^2 T}\right). \tag{23}$$

Although (22) cannot be evaluated in closed form, implicit differentiation reveals that the equity value need not be a monotone increasing function of the value of the firm—plots of the value of the shareholder's equity as a function of X are similar in appearance to Figure 4. This result has the intriguing implications that shareholders in a nearly or slightly bankrupt firm may have an incentive to take on negative NPV projects in order to drive down the value of the firm. The intuition for this surprising result is similar to that described for simple writer-extendible calls—the stockholders of the firm may prefer to have the loan maturity extended rather than be in a situation where the value of the firm is equal to or only slightly more than the face amount of the debt.

There are many other types of financial contracts that include flexible writer-extendible options. One example that is similar to that described above is international lending. In this case, the lender has a short call on the foreign exchange earnings of the debtor nation. However, unlike the previous example, there is no underlying collateral securing the debt. Thus, in the event of a default (an out-of-the-money expiration), the associated penalty to the lender could be catastrophic even if the lender received some form of a government bailout. This penalty would provide a strong incentive to the lender to renegotiate or reschedule payments on the defaulting debt over an extended period rather than to pursue other less profitable approaches of recovery. Note that many financial contracts which permit a grace period, a rescheduling of payments, a redemption interval, etc., could also be modeled using writer-extendible options.

V. Conclusion

We have derived closed-form expressions for the prices of calls and puts that are extendible by either the optionholder or the option writer. The results have broad applicability because many different types of financial contracts and contingent claims incorporate options with extendible maturities. For example, we have shown how these results can be applied to value real estate options, warrants,

[22] See Cox and Ross (1976) for a discussion of the risk-neutral density.

extendible junk bonds, American call options where the underlying asset pays stochastic dividends, shared-equity mortgages, lease contracts with an extendible option to purchase, international debt, and stock in levered firms with liquidation costs.

In addition, we have shown that the properties of extendible options can be quite different from those of conventional options. For example, writer-extendible calls need not be monotone increasing functions of the value of the underlying asset or of the volatility of returns on the underlying asset. This illustrates that the flexibility over the timing of cash flows providing by an explicit or implicit extension privilege can have dramatic effects on the properties of contingent claim prices.

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