Invest or Exit? Optimal Decisions in the Face of a Declining Profit Stream

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October 22, 2007

Abstract

Even in the face of deteriorating and highly volatile demand, firms often invest in rather than discard aging technologies. In order to study this phenomenon, we model the firm's profit stream as a Brownian motion with negative drift. At each point in time, the firm can continue operations, or it can stop and exit the project. In addition, there is a one-time option to make an investment which boosts the project's profit rate. Using stochastic calculus, we show that the optimal policy is characterized by three thresholds. There are investment and exit thresholds before investment, and there is a threshold for exit after investment. We also effect a comparative statics analysis of the thresholds with respect to the drift and the volatility of the Brownian motion. When the profit boost upon investment is sufficiently large, we find a counter-intuitive result: an increase in volatility induces the firm to invest earlier.

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1 Introduction

The computer disk drive industry underwent a series of disruptive architectural innovations (Christensen 1992). Until the mid-1970's, 14-inch hard disk drives dominated the mainframe computer disk drive market. Between 1978 and 1980, several new entrants introduced 8-inch disk drives which were initially sold to minicomputer manufacturers because their recording capacity was too small and the cost per megabyte was too high for mainframe computers. As the performance of 8-inch drives kept improving, the entrants quickly encroached upon the mainframe computer market. By the mid-1980's, 8-inch drives dominated the mainframe market and rendered 14-inch drives obsolete. Nevertheless, among the dozen or so established manufacturers of 14-inch drives, two thirds of them never introduced 8-inch drives. Instead, they continued to enhance the recording capacity of the extant 14-inch drives in order to appeal to the higher end mainframe market (Christensen 2000, p. 19). Eventually, all 14-inch drive manufacturers, except those that were vertically integrated, were forced out of the disk drive market. This pattern of industry-wide disruption emanating from the introduction of a successful new technology is a commonplace rather than an isolated incident; as such, it deserves serious attention. Even 8-inch drives were eventually superseded by 5.25-inch drives. Currently, the computer disk drive industry is in the process of yet another architectural transition, one from hard disk drives to flash solid state disks (used in USB stick drives).

This paper focuses upon the difficult investment and exit decisions of a firm facing a declining profit stream. With the onslaught of disruptive technological innovation, as in the example of the disk drive industry, a firm employing an extant technology faces a deteriorating profit stream due to declining demand and/or prices. Faced with a profit stream that has eroded, it might be optimal for the firm to cease operations and avoid recurring losses. On the other hand, if the erosion has not been too large, then it can be optimal for the firm to make an additional investment in the project. The pressing question is when, if ever, to invest and when to exit. Exit ought to occur when the current profit rate is sufficiently negative; a negative value of the profit rate, however, is not a sufficient condition to induce exit as the option to cease operations sometime in the future must be taken into account. Likewise, a firm must invest in its operations in a timely fashion before the desirable investment opportunity vanishes. In a highly volatile environment such as in the disk drive industry, however, it is difficult to calculate the optimal time to invest or exit because of the uncertainty in the future demand. After we obtain the optimal policy, we examine how increases in uncertainty affect the optimal policy.

In light of the declining demand, it seems counter-intuitive to invest in the current operation. However, in the example of the computer hard disk drive industry, the manufacturers of 14-inch disk drives continued investment even though they faced a deteriorating profit stream and, as it turned out, eventual displacement from the industry. Christensen (2000) finds such examples in the mechanical excavator industry and the steel mill industry as well.

The two salient features of our model are the possibility of exit and a declining stochastic profit stream. In particular, the firm can exit at any point in time, and we model the firm's uncertain profit stream as a Brownian motion X_t with drift μ and volatility σ where both μ and σ are time-independent constants known to the firm. Of course, the drift μ is the average rate of change in the profit rate, and the volatility σ measures the underlying uncertainty. Although the sign of the drift μ is unrestricted, we give special attention to the case in which μ is negative. With this representation, the firm's cumulative profit is the time-integral of the Brownian motion. The investment and exit decision rules of the firm are stopping times for the Brownian motion, and we utilize the well-known machinery of stochastic differential equations (Oksendal 2003) to find the optimal stopping times.

In Sec. 3, we present the basic model in which investment is not possible. At each point in time, the firm must decide whether to continue operations or irrevocably exit the project. The firm seeks to maximize its expected discounted cumulative profit by selecting the optimal time τ at which to exit, where τ is a stopping time for the Brownian motion. In Sec. 4, we show that the optimal policy is a threshold rule: it is optimal to continue operations until the profit rate X_t falls below a critical threshold ξ_0, at which time it is optimal to exit. The closed-form expression for ξ_0 is a decreasing function of μ and σ, and it reveals that ξ_0 is negative.

In Sec. 5, we extend the basic model to include a one-time opportunity to invest in improving the extant technology: at each point in time, the firm can (1) continue operations, (2) stop and irrevocably exit the project, or (3) invest in the operations. The investment increases both the current profit rate and the drift of the profit stream by known quantities. In view of the investment opportunity, the firm's policy is specified by three stopping times. The firm must specify when to exit and when to invest while the investment option is still available. If the firm already has made the investment, then the firm must decide when to exit. Each stopping time is characterized by a threshold. If investment has not been made, it is optimal to exit whenever the profit rate falls below a threshold ξ_E, and it is optimal to invest if the profit rate rises above a second threshold ξ_I. When the current profit rate is between ξ_E and ξ_I, it is optimal to maintain the *status quo*: continue operations but do not invest. Because there is only one opportunity to invest, after investment, the firm's decision problem reduces to that of the basic model, albeit with different drift: after investment, the firm exits when the current profit rate drops below a third threshold ξ_1.

After finding the optimal policy, we effect a comparative statics analysis of the thresholds ξ_I and ξ_E with respect to μ and σ. Although it is intuitively clear that the optimal policy is characterized by thresholds, the comparative statics analysis is not straightforward. In order to obtain ξ_I and ξ_E, we first need to solve an optimal stopping time problem with a reward which depends on the return from investment. The complication is that the return from investment in turn depends on both μ and σ because the firm will continue operations prior to eventual exit. Nevertheless, we have been able to effect a comparative statics analysis using a power-expansion method without resorting to a numerical analysis.

Regarding the comparative statics of the threshold for investment (ξ_I), we might be able to derive some useful insights from real options theory. Real options theory has shown that, under certain mild conditions, it is optimal to wait longer before making an irrevocable investment if the volatility of the underlying asset increases (Dixit 1992). Waiting and observing the evolution of the value of the asset enables the investor to avoid the downturn risk and take advantage of the upturn potential. In accord with this intuition, we anticipate that ξ_I increases in σ because the upturn potential of the profit stream increases in σ. Indeed, if the boost in the profit rate upon investment is small enough, then ξ_I increases in σ as expected. Surprisingly, if the boost is sufficiently large, then ξ_I decreases in σ. This seemingly counter-intuitive result obtains because the return from investment rapidly increases in σ due to the post-investment option to exit. In the operations context, this comparative statics result offers cautionary advice against blindly following the intuition inherited from real options theory. See, for example, Bollen (1999) who shows that if the product life cycle (demand dynamics) is ignored, then the conventional real-option technique tends to undervalue capacity contraction and overvalue capacity expansion.

This paper is organized as follows. We review related literature in Sec. 2 and formally present our basic model in Sec. 3. The analysis of the basic model without an investment opportunity is performed in Sec. 4. The basic model is extended to include one investment opportunity in Sec. 5; Sec. 5.2 is devoted to the analysis of the extended model. Lastly, we effect the comparative statics of the thresholds in Sec. 5.3.

2 Related Literature

There is rich literature on technology and process adoption. (See, for example, Bridges et al. 1991 for a review.) In an early paper which formulates technology adoption as an investment problem, Barzel (1968) uses the net-present-value approach to obtain the optimal timing of a one-time investment in adoption of technology when the future profit stream is deterministic. In the context of process improvement, Porteus (1985) uses the EOQ model to examine the economic trade-offs between the cost of investment which reduces the setup cost and the benefit from the reduced setup cost: the optimal policy is to invest if and only if the sales rate is above a threshold. Porteus (1986) extends this work by examining a model in which lower setup costs lead to improved quality control (lower defect rate).

An objective of the current paper is to obtain investment and exit policy under uncertainty. Many papers have modeled technology adoption as a stopping time problem. (See, for example, Hoppe 2002 for a survey of literature.) For example, Balcer and Lippman (1984) study the optimal time to adopt the best currently available technology when multiple adoptions are allowed. In their model, the *timing* and the *value* of future innovations is uncertain although the profitability of the currently available technology is known. They show that it is optimal to adopt the best currently available technology if the technological lag exceeds a threshold which depends upon the multi-dimensional state: the elapsed time since last innovation and the pace (rapidity) of technological progress.

There is substantial literature on Bayesian models of investment and exit. Jensen (1982) develops a decision-theoretic framework of technology adoption when the profitability of the technology is uncertain. In his model, a firm considers adopting a technology which is either a success or a failure. The probability θ of success is unknown to the firm, but it takes one of two known values. In each period, the firm costlessly observes a Bernoulli random variable with the parameter θ, updates its belief regarding the value of θ, and decides whether to adopt the technology. Adoption of a successful (unsuccessful) technology produces a positive (negative) profit, and all returns are discounted. The optimal policy is a threshold rule with respect to the posterior probability that θ takes the higher value: adopt the technology when this probability is sufficiently high. McCardle (1985) extends Jensen's work by studying a model where it is costly to acquire information on the uncertain profitability of the technology. In McCardle's model, the firm must pay to observe Bernoulli random variables which allow the firm to update its belief concerning the technology's profitability (which can assume a continuum of values rather than two as per Jensen). At each point in time, the firm can continue acquiring information, irreversibly adopt the technology, or exit. The optimal decision rule is characterized by two (upper and lower) thresholds with respect to the expected profitability. Later, Mamer and McCardle (1987) extend McCardle's work by studying the same model with competition which is either substitute or complementary, and they obtain Nash equilibria.

Exit policies have been also studied via a Bayesian decision-theoretic approach. Ryan and Lippman (2003) study optimal exit policy under imperfect information on the profit stream. They model the cumulative profit as a Brownian motion in which the drift in cumulative profit is one of two known constants; the higher drift is positive and the lower drift is negative. The decision-maker updates his belief about the value of the drift by observing the realized profit at each point in time. Using stochastic calculus, they show that it is optimal to exit when the posterior probability that the drift is negative is high. Ryan and Lippman (2005) extend this model by allowing the drift to drop to the lower value after an unobservable exponential time. By observing the realized profit, the decision-maker updates his posterior probability that the profit stream has dropped. The optimal exit decision in this model is also a threshold rule with respect to the posterior probability.

One focus of our paper is the impact of uncertainty on the investment and exit decisions. Dixit (1992, p. 108) points out that, as uncertainty increases, it is optimal to wait longer before investment if (1) the investment is irreversible, (2) the uncertainty regarding the investment is being resolved gradually in time, and (3) the investment can be flexibly postponed. In this vein, McDonald and Siegel (1986) study investment in an asset whose value and price evolve as geometric Brownian motion. They find that the optimal policy is a threshold rule with respect to the ratio of the value to the price of the asset. Moreover, the investment threshold increases in the volatility: it is optimal to postpone investment longer as the uncertainty increases.

A number of papers address the effect of uncertainty on technology adoption using the real options approach. Essentially, they confirm the conventional intuition regarding the value of waiting. Farzin et al. (1998) study the optimal time to irreversibly switch to new technology when the value and the arrival date of future improvements are uncertain. In their model, the improvement in the value of the currently available technology follows a compound Poisson process. They allow multiple investments in technology; again, the optimal policy is a threshold rule. In particular, they find that the pace of adoption is slower with the real-option method than with the suboptimal net-present-value method. Alvarez and Stenbacka (2001) also use the real options approach to study the optimal time to adopt a technology with an opportunity for improvement after adoption. Once the firm adopts the technology, it receives a revenue stream which evolves stochastically over time: at an exponential time, an improved technology becomes available to the firm. They show that increased market uncertainty (volatility) increases the real-option value of adopting the initial technology.

The real options method has also been applied to exit in a duopoly game when the profit stream is stochastic. Fine and Li (1986) find a Nash equilibrium in stopping times in their discrete-time duopoly game of exit from a market with declining stochastic demand. Murto (2004) studies a similar duopoly exit game in an industry in which the declining demand follows a geometric Brownian motion; he obtains Markov-perfect equilibria. Although these two papers analyze a duopoly model, they also consider the exit problem of a monopolist which is similar to our basic model. Their focus, however, is on the strategic interaction rather than on the uncertainty.

In addition to the uncertainty in the profit stream, there is a complicating but salient feature in our model: exit is possible after investment. Among the papers that include this feature, McDonald and Siegel (1985) study the valuation of a manufacturing firm facing a stochastic price for its output product using option pricing techniques. In their model, the product price is a geometric Brownian motion, and the firm can shutdown and re-open its plant without cost at any point in time. In contrast, Dixit (1989) considers fixed cost of entry and exit. In his model, the firm can enter and exit the industry as many times as the firm wishes, and the profit stream is a geometric Brownian motion. He shows that it is optimal to invest if the profit rate is above an upper threshold and exit if it is below a lower threshold. He performs a numerical comparative statics analysis and finds that the upper (lower) threshold increases (decreases) in the volatility. In his model, the investment (entry) decision can be exercised only by an inactive firm; of course, the exit decision can be exercised only by active firms. Our paper studies investment and exit decisions in a quite different model: the firm has one opportunity to invest in its operations while being active in the industry, and it can exit at any point in time. Moreover, our comparative statics results are analytical.

In the literatures on technology adoption and on exit, there is a paucity of work on investment when the firm faces a declining profit stream. To our knowledge, the current paper is the first to study the impact of uncertainty on investment in an on-going project with an exit option available both before and after an investment.

3 The Basic Model

Consider a manufacturing firm whose product is produced with an aging technology or process. Because of obsolescence, its profit stream is in decline (perhaps because a substitute product produced with a new technology is encroaching upon the market). At any point in time, the firm can stop the project by permanently closing its production plant.

The firm, seeking to maximize the expected discounted value of its profit stream over an infinite horizon, must determine the best time to cease operations and exit the market. The firm's profit rate at time t is a random variable X_t where $\{X_t : t \geq 0\}$ is a stochastic process with continuous sample paths whose law of motion we will specify shortly. We refer to $\{X_t : 0 \leq t \leq \tau\}$ as the firm's *profit stream* where the stopping time $\tau \leq \infty$ is the time of exit. Even when $E[X_t]$ is strictly decreasing in t, there is a positive probability that $X_{t+u} > X_t$ for some time $u > 0$. Consequently, it may not be optimal to exit the industry the first time X_t hits zero.

We model the firm's profit stream as a Brownian motion with constant drift μ and volatility σ. Specifically, let X_t denote the profit at time t with $X_t = X_0 + \mu t + \sigma B_t$ where $\{B_t : t \geq 0\}$ is a one-dimensional standard Brownian motion, so the profit stream has constant drift μ and constant volatility σ. We pay particular attention to the case $\mu < 0$ because our main focus is modeling a declining profit stream. If the firm begins operations at time t_1 and exits at time t_2, the discounted value of its profit stream is $\int_{t_1}^{t_2} e^{-\alpha t} X_t dt$, where α is the discount rate.

To illustrate, suppose that the demand D_t per unit time for the firm's product is a Brownian motion with drift μ/p, where p is the sales price per unit, and let c be the fixed cost of operation per unit time. Then the relationship between the demand and the profit stream is linear:

$$X_t = pD_t - c\,. \tag{1}$$

4 Analysis of the Basic Model

In our basic model, at each point in time, the firm must elect either to continue operations or to exit irrevocably. Because there is no investment opportunity in the basic model, the firm seeks a stopping time τ which maximizes

$$E^x[\int_0^\tau X_t e^{-\alpha t} dt]\,, \tag{2}$$

where $E^x[\cdot] \equiv E[\cdot|X_0 = x]$, the expectation conditioned on $X_0 = x$. (To be more precise, $\{X_t : t \geq 0\}$ is a one-dimensional Brownian motion adapted to a filtration $\{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, P)$. The random variable τ is an element of $\mathcal{T}$, the set of all non-negative stopping times with respect to the filtration $\{\mathcal{F}_t\}$. When possible, we skip over measure-theoretic niceties.)

The objective function in Eq. (2) has no time-dependence other than through the process X_t and the discount factor $e^{-\alpha t}$; hence, we can show directly (or use the argument of Oksendal 2003, p. 220) that the optimal policy is stationary: there is a set $D \subset \mathbb{R}$ such that it is optimal to continue operations as long as $X_t \in D$ and stop when $X_t \notin D$. The set D is called a *continuation set* D. Throughout the paper, we let τ_A denote the *first exit time of the process* X_t *from the measurable set* A:

$$\tau_A \equiv \inf\{t > 0 : X_t \notin A\}\,.$$

Define $R_D(x)$, the expected return when using the stopping time τ_D, by

$$R_D(x) = E^x[\int_0^{\tau_D} e^{-\alpha t} X_t dt]\,, \tag{3}$$

and let $V_0(x)$ denote the firm's optimal return where $X_0 = x$. Because there is an optimal policy which is stationary,

$$V_0(x) = \sup_D R_D(x) = R_{D^*}(x)$$

where D^* is the optimal continuation set. The optimal continuation set is also determined by $D^* = \{x : V_0(x) > 0\}$, so D^* is an open set because $V_0(\cdot)$ is continuous. We elect not to include the boundary of D^* in the continuation set.

Immediately below, we show that D^* is an interval of the form (ξ_0, ∞): there is a threshold ξ_0 such that it is optimal to exit immediately when $X_t \leq \xi_0$ while it is optimal to continue operations as long as $X_t > \xi_0$. Of course, $V_0(\cdot)$ is strictly increasing on (ξ_0, ∞). Interestingly, we present a closed form solution for ξ_0 and easily demonstrate that $\xi_0 < 0$. By relegating some of the technical details to Appendix B, our proof does not require a background in stochastic calculus.

Proposition 1: There is a number $\xi_0 < 0$ such that $D^* = (\xi_0, \infty)$, where

$$\xi_0 = -\frac{\mu}{\alpha} - \frac{\sigma^2}{\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}}\,. \tag{4}$$

Moreover, the optimal return function is $V_0(x) = 0$ if $x \leq \xi_0$ and

$$V_0(x) = \frac{1}{\alpha}\left\{x + \frac{\mu}{\alpha} + \frac{\sigma^2}{\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}} \exp[\frac{-\mu - \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2}(x - \xi_0)]\right\}, \text{ if } x > \xi_0 . \tag{5}$$

Proof. We prove Proposition 1 in three steps: (i) We show that the optimal continuation set is of the form (ξ, ∞) where $\xi \leq 0$, (ii) we explicitly construct the return function $R_{(\xi,\infty)}(x)$ for any ξ, and (iii) we obtain $\xi = \xi_0$ that maximizes $R_{(\xi,\infty)}(x)$ and show that ξ_0 is a negative number.

(i) We begin by showing that the optimal continuation set D^* is of the form (ξ, ∞). We first claim that $(0, \infty) \subset D^*$. If not, there is $x > 0$ not in D^* so that $V_0(x) = 0$. However, with $A = (x/2, \infty)$, we see that

$$R_A(x) = E^x[\int_0^{\tau_A} X_t e^{-\alpha t} dt] > \frac{x}{2\alpha}(1 - E e^{-\alpha\tau_A}) > 0 = V_0(x) ,$$

a contradiction.

Next, we claim that $D^* = (\xi, \infty)$ for some $\xi \leq 0$. If $V_0(x) = 0$ for all $x < 0$, then $D^* = (0, \infty)$. If $V_0(x) > 0$ for some $x < 0$, then $(x, 0] \subset D^*$; otherwise, there is a $y \notin D^*$ such that $x < y \leq 0$. Using the fact that $\{X_t : t \geq 0\}$ has continuous sample paths, we have

$$V_0(x) = E^x[\int_0^{\tau_{D^*}} X_t e^{-\alpha t} dt] \leq y E^x[\int_0^{\tau_{D^*}} e^{-\alpha t} dt] \leq 0 < V_0(x) ,$$

a contradiction. Thus, $D^* = (\xi, \infty)$ where $\xi = \inf\{x : V_0(x) > 0\} \leq 0$.

(ii) Consider the continuation set $D = (\xi, \infty)$. In Appendix B, we show that $R_D(x)$, defined in Eq. (3), satisfies the partial differential equation (PDE)

$$\left(\partial_t + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2\right)[e^{-\alpha t} R_D(x)] = -e^{-\alpha t} x \quad \text{for } x \in D, \tag{6}$$

where we use the abbreviated notation for partial derivatives: $\partial_x f \equiv \partial f / \partial x$. Appendix B also establishes that $R_D(\cdot)$ satisfies the linear bound

$$R_D(x) \leq (1 + |x|)\alpha^{-1} + (|\mu| + \sigma^2)\alpha^{-2} . \tag{7}$$

Finally, because the profit rate is identically zero after exit (which is an irrevocable decision), $R_D(x) = 0$ for $x \leq \xi$.

It is easy to verify that the function $f(\cdot)$ given below satisfies Eq. (6) and (7):

$$f(x) = \alpha^{-1}(x+\mu/\alpha) - \alpha^{-1}(\xi+\mu/\alpha)\exp\left[\frac{-\mu-\sqrt{\mu^2+2\alpha\sigma^2}}{\sigma^2}(x-\xi)\right] \qquad \text{for } x > \xi,$$

and $f(x) = 0$ for $x \le \xi$. Moreover, it is well-known that there is a *unique* solution to the kind of PDE given in Eq. (6) satisfying a linear bound as per Eq. (7) and the boundary condition $R_D(\xi) = 0$ (see Arfken 1985, Chapter 8). Therefore, $R_{(\xi,\infty)}(x) = f(x)$.

(iii) Setting $dR_{(\xi,\infty)}(x)/d\xi = 0$, it is straightforward to verify that $R_{(\xi,\infty)}(x)$ achieves its maximum for all $x \in \mathbb{R}$ when $\xi = \xi_0$ as given by Eq. (4).

To show that $\xi_0 < 0$, note that $\mu + \sqrt{\mu^2+2\alpha\sigma^2} > 0$ and observe that

$$\begin{aligned}\alpha(\mu+\sqrt{\mu^2+2\alpha\sigma^2})\xi_0 &= -\mu(\mu+\sqrt{\mu^2+2\alpha\sigma^2}) - \alpha\sigma^2 \\ &= -(\mu^2+\alpha\sigma^2) - \mu\sqrt{\mu^2+2\alpha\sigma^2}\,,\end{aligned}$$

which is negative because $\mu^2+\alpha\sigma^2 > -\mu\sqrt{\mu^2+2\alpha\sigma^2}$. ■

It is intuitively clear that the firm will exit if its profit rate has deteriorated below some threshold, but the fact that the threshold is negative is not obvious. The reason $\xi_0 < 0$ is that there is value in waiting before taking an irrevocable action: even if $\mu < 0$ and the current profit rate is slightly negative, it is possible for the profit rate to turn positive in the future. If the profit stream were deterministic and monotonically decreasing, then it would be optimal to exit when the profit rate hits zero. This intuition regarding the value of waiting is consistent with the fact that ξ_0 increases to 0 as $\sigma \to 0$, which follows from Eq. (4) when $\mu < 0$.

The value of remaining in business while incurring losses has been demonstrated in practice. Apple Inc. was hemorrhaging money in 1996 and 1997 with losses of \$0.8 and \$1 billion, respectively. Michael Dell, the CEO of Dell Computer, remarked "What would I do? I'd shut it down and give the money back to the shareholders." (CNET News.com, October 6, 1997.) However, Apple did not exit/shut down; instead, its fortunes improved dramatically, and its split-adjusted stock price increased from \$5.48 on October 6, 1997 to \$172.75 on October 17, 2007.

Because we have a closed-form expression for ξ_0, it is straightforward to obtain its comparative statics. For convenience, we first define

$$\gamma_p = (-\mu+\sqrt{\mu^2+2\alpha\sigma^2})/\sigma^2 \qquad \text{and} \qquad \gamma_n = (-\mu-\sqrt{\mu^2+2\alpha\sigma^2})/\sigma^2\,. \tag{8}$$

The constants γ_p and γ_n are the two roots of the quadratic equation:

$$-\alpha+\mu\gamma+\frac{1}{2}\sigma^2\gamma^2=0\,.$$

The quadratic equation is derived from the second-order partial differential equation applied to the exponential function $e^{-\alpha t+\gamma x}$: $(\partial_t+\mu\partial_x+\frac{1}{2}\sigma^2\partial_x^2)\exp(-\alpha t+\gamma x)=0$.

Corollary 2: The threshold ξ_0 decreases in μ and σ. When $\mu<0$, $\xi_0\uparrow 0$ as $\sigma\to 0$.

Proof. From Eqs. (4) and (8), we have

$$\partial_\mu\xi_0 = -\frac{1}{\alpha}+\frac{1}{\gamma_n^2\sigma^2}+\frac{\mu}{\gamma_n^2\sigma^2\sqrt{\mu^2+2\alpha\sigma^2}}\,, \tag{9}$$

$$\partial_{\sigma^2}\xi_0 = \frac{-\mu-\sqrt{\mu^2+2\alpha\sigma^2}}{\gamma_n^2\sigma^4}+\frac{\alpha}{\gamma_n^2\sigma^2\sqrt{\mu^2+2\alpha\sigma^2}}\,. \tag{10}$$

After some algebra, we can show that

$$-\alpha^{-1}\gamma_n^2\sigma^2\sqrt{\mu^2+2\alpha\sigma^2}+\sqrt{\mu^2+2\alpha\sigma^2}+\mu<0\,, \tag{11}$$

$$-\mu\sqrt{\mu^2+2\alpha\sigma^2}-(\mu^2+2\alpha\sigma^2)+\alpha\sigma^2<0\,. \tag{12}$$

The inequalities of Eqs. (11) and (12) are independent of the sign of μ. Hence, both partial derivatives of ξ_0, Eqs. (9) and (10), are negative. Lastly, employ L'Hospital's rule to verify $\lim_{\sigma\to 0}\xi_0\uparrow 0$. ■

The comparative statics of ξ_0 is tightly linked to that of $V_0(\cdot)$. Because $V_0(\cdot)$ is non-decreasing, the threshold is determined by $\xi_0=\inf\{x:V_0(x)>0\}$. By this relation, if the value of the return function $V_0(\cdot)$ is larger (smaller), then ξ_0 is lower (higher).

Corollary 3: The optimal return function $V_0(\cdot)$ increases in σ and μ for $x>\xi_0$.

Proof. We first examine the dependence of $V_0(\cdot)$ on σ. Let $V_0(x;\sigma^2)$ denote the optimal return function and let T_{σ^2} denote the optimal exit time when the volatility of X_t is σ. We introduce $\bar{B}_t$, a one-dimensional standard Brownian motion which is independent of the process X_t. For any $\sigma_1>0$, we have

$$V_0(x;\sigma^2)=E^x\int_0^{T_{\sigma^2}}X_te^{-\alpha t}dt=E^x\int_0^{T_{\sigma^2}}(X_t+\sigma_1\bar{B}_t)e^{-\alpha t}dt$$

$$\leq E^x \int_0^{T_{\sigma^2+\sigma_1^2}} (X_t + \sigma_1 \bar{B}_t) e^{-\alpha t} dt = V_0(x; \sigma^2 + \sigma_1^2) \,.$$

Hence, $V_0(x)$ is non-decreasing in σ^2. In fact, it is easy to show that $V_0(x)$ is strictly increasing in σ. From the closed-form expression of $V_0(x)$ and the inequality

$$\partial_{\sigma^2}\gamma_n = \frac{\mu}{\sigma^4} + \frac{\sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^4} - \frac{\alpha}{\sigma^2\sqrt{\mu^2 + 2\alpha\sigma^2}} > 0 \,,$$

we can directly calculate

$$\frac{\partial V_0(x)}{\partial \sigma^2} = -(\alpha\gamma_n)^{-1}\partial_{\sigma^2}\gamma_n(x - \xi_0)e^{\gamma_n(x-\xi_0)} > 0 \quad \text{for all}\, x > \xi_0 \,,$$

in agreement with $\partial_{\sigma^2}\xi_0 < 0$.

Similarly, let $V_0(x;\mu)$ denote the optimal return function and let T_μ denote the optimal exit time when the drift of X_t is μ. Let $x > \xi_0$ so that $T_\mu > 0$. For any $\delta > 0$,

$$\begin{aligned} V_0(x;\mu) = E^x \int_0^{T_\mu} X_t e^{-\alpha t} dt &< E^x \int_0^{T_\mu} (X_t + \delta t) e^{-\alpha t} dt \\ &\leq E^x \int_0^{T_{\mu+\delta}} (X_t + \delta t) e^{-\alpha t} dt = V_0(x; \mu + \delta) \,. \end{aligned} \tag{13}$$

Hence, $V_0(x;\mu)$ is increasing in μ for $x > \xi_0$ in agreement with the comparative statics result $\partial\xi_0/\partial\mu < 0$. ■

As σ increases, there is more noise in the profit stream, so there is a larger upturn potential as well as a larger downturn risk. However, the firm can take advantage of the upturn potential while avoiding downturn risk by exit. Hence, the return function increases in σ. Because an increase in μ improves the profit stream X_t, the return $R_D(\cdot)$ increases for each continuation set D.

Lastly, we examine the impact of adding a lump sum salvage value s receivable at the time of exit. If plant and equipment are sold upon exit, then we anticipate $s > 0$. However, if there is employee severance or liabilities associated with decommissioning of the business, then $s < 0$.

Proposition 4: Let $V(\cdot; s)$ denote the optimal return function when s is the salvage value. Then

$$V_0(x;s) = s + V_0(x - \alpha s) \,,$$

and the exit threshold is $\xi(s) = \xi_0 + \alpha s$.

Proof. Because $se^{-\alpha\tau} = s - \int_0^\tau \alpha s e^{-\alpha t} dt$,

$$V_0(x;s) = E^x[\int_0^\tau X_t e^{-\alpha t} dt + s e^{-\alpha\tau}] E^x = s + E^x[\int_0^\tau (X_t - \alpha s) e^{-\alpha t} dt]$$
$$= s + E^{x-\alpha s}[\int_0^\tau X_t e^{-\alpha t} dt] = s + V_0(x - \alpha s)\ .$$

Because $V_0(x;s)$ is increasing in s, the exit threshold is

$$\xi(s) = \inf\{x : V_0(x;s) > s\} = \xi_0 + \alpha s\,.$$

■

In light of Proposition 4, in the remainder of the paper, we proceed with $s = 0$ without loss of generality.

5 The Model with One Investment Opportunity

In this section, we consider the possibility of a once-in-a-lifetime investment. For instance, manufacturers of 14-inch disk drives can, despite the writing on the wall, improve the performance (recording capacity) of 14-inch drives in order to immediately boost demand in the higher-end mainframe computer market (Christensen 2000, p. 19). Of course, exit is inevitable when $\mu < 0$. The sign of μ is unrestricted except in Sec. 5.3.

For analytical tractability, our model allows only one investment opportunity. As suggested by Fine and Porteus (1989), in practice, the firm might have multiple opportunities for gradual improvement in the technology/process. The impact of multiple investment opportunities is beyond the scope of this paper.

5.1 The Model

We now include a one-time opportunity to implement an innovation which improves the quality of the product or the process. The implementation cost is $k > 0$. If the quality of the product improves, then the demand for the product increases; moreover, the demand declines more slowly. Specifically, the investment boosts the current profit rate by b and increases the drift by δ. In terms of Eq. (1), investment induces an increase of b in pD_t (or, equivalently, a decrease of b in c) and an increase of

δ in $p \cdot dD_t/dt$. If the firm invests at time τ, then the improved profit stream follows the process

$$Y_t = X_t + \delta(t-\tau) + b\,, \quad \text{for } t > \tau$$

so that $dY_t = (\mu+\delta)dt + \sigma dB_t$.

We examine the conditions under which it is never optimal to invest. Define

$$g \equiv \alpha\Big(\int_0^\infty (b+\delta t)e^{-\alpha t}dt - k\Big) = b + \delta/\alpha - k\alpha \tag{14}$$

so that g/α is the net discounted gain from investment if exit never occurs.

Proposition 5: Let I be the set of states (profit rates) from which it is optimal to invest immediately. Then I is non-empty if and only if $g > 0$.

Proof. If $g \le 0$, we claim that $D^* = (\xi_0, \infty)$ and $V_1(x) = V_0(x)$: it is never optimal to invest. In order to show that it is never optimal to invest, we compare the return functions of two candidate policies. The first candidate policy is to invest at some stopping time τ_I and exit at another stopping time τ_E; after investment, the policy is to exit at a third stopping time τ_1. The return function of this policy is

$$R_1(x) = E^x\Big[\int_0^\tau X_t e^{-\alpha t}dt + \mathbf{1}_{\{\tau_I<\tau_E\}}\Big(\int_\tau^{\tau_1}(b+\delta(t-\tau)+X_t)e^{-\alpha t}dt - ke^{-\alpha\tau}\Big)\Big]\,,$$

where $\tau \equiv \tau_I \wedge \tau_E$. For convenience, we define $\tau_1 = \tau = \tau_E$ if $\tau_E < \tau_I$. The second candidate policy is to never invest and to exit at time $\tau_0 = \inf\{t>0 : X_t \le \xi_0\}$. Its return function is $R_2(x) = E^x[\int_0^{\tau_0} X_t e^{-\alpha t}dt]$. Because τ_0 is the optimal time to exit in the absence of investment, we have the inequality $E^x[\int_0^T X_t e^{-\alpha t}dt - \int_0^{\tau_0} X_t e^{-\alpha t}dt] \le 0$ for any stopping time T. In particular, this inequality holds for $T = \tau_1$. Thus,

$$\begin{aligned}
R_1(x) - R_2(x) =& E^x\Big[\mathbf{1}_{\{\tau_I<\tau_E\}}e^{-\alpha\tau}\Big(\int_0^{\tau_1-\tau}(b+\delta t)e^{-\alpha t}dt - k\Big)\Big] + E^x\Big[\int_0^{\tau_1} X_t e^{-\alpha t}dt - \int_0^{\tau_0} X_t e^{-\alpha t}dt\Big] \\
\le& E^x\Big[\mathbf{1}_{\{\tau_I<\tau_E\}}e^{-\alpha\tau}\Big(\int_0^{\tau_1-\tau}(b+\delta t)e^{-\alpha t}dt - k\Big)\Big] \le z\Big[\int_0^\infty (b+\delta t)e^{-\alpha t}dt - k\Big] \\
=& z g/\alpha \le 0\,,
\end{aligned}$$

where $z \equiv E^x[\mathbf{1}_{\{\tau_I<\tau_E\}}e^{-\alpha\tau}] \ge 0$; it is *never* optimal to invest if $g \le 0$.

Assume $g > 0$, and suppose that it is never optimal to invest. By Proposition 1, the optimal continuation set is $D^* = (\xi_0, \infty)$ and the optimal return function is $V_0(x)$. We compare $V_0(x)$ with the expected return from immediate investment, $V_0^+(x+b)-k$. Let $\tau_0 = \inf\{t>0 : X_t < \xi_0\}$ denote the optimal exit time when there is no investment opportunity, and fix $x > \max\{\xi_1 - b, \xi_0\}$. Using integration by parts, we have

$$\begin{aligned} V_0^+(x+b)-k-V_0(x) &\geq E^x[\int_0^{\tau_0}(X_t+\delta t+b)e^{-\alpha t}dt - k - \int_0^{\tau_0} X_t e^{-\alpha t}dt] \\ &= -E^x[(b/\alpha+\delta/\alpha^2+\delta\tau_0/\alpha)e^{-\alpha\tau_0}] + g/\alpha\,. \end{aligned}$$

It suffices to show that $E^x[e^{-\alpha\tau_0}]$ and $E^x[\tau_0 e^{-\alpha\tau_0}]$ converge to zero as $x\to\infty$. For fixed ξ, we let $\tau = \inf\{t>0 : X_t<\xi\}$ denote the exit time from the interval (ξ,∞). Define $f(x,t)\equiv E^x[e^{-\alpha\tau}]e^{-\alpha t}$, then by Eq. (25) and by the same argument given in Appendix B for unbounded continuation sets, we have $\mathcal{L}f(x,t)=0$. In addition, $f(x,t)$ satisfies the boundary condition $f(\xi,0)=1$ and the boundedness condition $f(x,t)\leq 1$. Then it is easy to verify that $f(x,t)=\exp[\gamma_n(x-\xi)-\alpha t]$ for $x>\xi$. Thus, $E^x[e^{-\alpha\tau}] = e^{\gamma_n(x-\xi)}\to 0$ as $x\to\infty$.

Because $\tau e^{-\alpha\tau}$ is a bounded function of τ, we can interchange the order of expectation and differentiation to obtain $E^x[\tau e^{-\alpha\tau}] = -dE^x[e^{-\alpha\tau}]/d\alpha = -\partial_\alpha\gamma_n(x-\xi)e^{\gamma_n(x-\xi)}$. Replacing ξ with ξ_0 and τ with τ_0, it follows that $E^x[(b/\alpha+\delta/\alpha^2+\delta\tau_0/\alpha)e^{-\alpha\tau_0}]$ is arbitrarily small for sufficiently large values of x. Hence, $V_0^+(x+b)-k-V_0(x)>0$ for sufficiently large x, contradicting the assumption $I=\emptyset$. ■

In light of Proposition 5, we assume $g>0$ for the remainder of the paper.

In the spirit of backward induction, we first examine the optimal policy after the firm has already made an investment. Because there is only one opportunity for investment, the post-investment problem reduces to that of Sec. 4 except that the drift of the profit stream has changed. We define

$$\begin{aligned} \mu^+ &= \mu+\delta\,, \\ \lambda &= (-\mu^+ - \sqrt{(\mu^+)^2+2\alpha\sigma^2})/\sigma^2\,, \\ \xi_1 &= -\mu^+/\alpha+\lambda^{-1}\,, \end{aligned} \tag{15}$$

where μ^+ and ξ_1 are the post-investment drift and the optimal exit threshold, respectively. Hence, after investment, the expected return as a function of the initial profit rate x is given by

$$V_0^+(x) = \begin{cases} \alpha^{-1}\{x+\mu^+/\alpha-\lambda^{-1}\exp[\lambda(x-\xi_1)]\}, & \text{for } x>\xi_1 \\ 0\,, & \text{otherwise} \end{cases}\,.$$

We note that $\xi_1 < \xi_0$ because ξ_0 decreases in μ.

Prior to investment, the firm needs to find the optimal stopping time τ at which to invest or to exit, whichever action results in a better payoff. If the firm invests at time τ, then its expected return starting at time τ is $V_0^+(X_\tau + b) - k$ because its expected cumulative profit stream after investment is $V_0^+(X_\tau + b)$ and the cost of investment is k. On the other hand, if the firm exits at time τ, then its return starting at time τ is 0. Hence, the firm receives the expected payoff of $\max\{V_0^+(X_\tau + b) - k, 0\}$ at time τ when it makes its investment or exit decision.

Let x^+ be the unique number which satisfies

$$V_0^+(x^+ + b) = k \quad . \tag{16}$$

(This definition uniquely determines x^+ because $V_0^+(x)$ is strictly increasing in x for all x such that $V_0^+(x) > 0$.) Then, at the optimal stopping time τ, it is optimal to exit if $X_\tau < x^+$ and invest if $X_\tau > x^+$ because $V_0^+(x+b) - k > 0$ if $x > x^+$ and $V_0^+(x+b) - k < 0$ if $x < x^+$. If the current profit rate X_t is x^+, then immediate investment and immediate exit both yield zero expected return. Appendix C shows that the optimal expected return is strictly positive when $X_t = x^+$, so it is not optimal to invest or exit immediately when $X_t = x^+$. Hence, $X_\tau \neq x^+$.

The optimal policy is stationary because neither the payoff $\max\{V_0^+(X_t + b) - k, 0\}$ nor the profit stream has any time-dependence other than through X_t and $e^{-\alpha t}$. Thus, we only have to consider a class of stopping times $\tau_D = \inf\{t > 0 : X_t \notin D\}$ expressed with respect to continuation sets D. We can express the objective function as

$$R_D(x) = E^x\left[\int_0^{\tau_D} e^{-\alpha t} X_t dt + e^{-\alpha \tau_D} h(X_{\tau_D})\right] , \tag{17}$$

where $h(\cdot)$ is the lump sum payoff defined by

$$h(x) = \max\{0, V_0^+(x+b) - k\} . \tag{18}$$

In this new representation, the firm's policy is to continue operations as long as $X_t \in D$ and to stop as soon as $X_t \notin D$, at which time the firm receives $h(X_t)$.

5.2 Analysis

Our objective is to find the optimal return function

$$V_1(x) = \sup_D R_D(x) \equiv R_{D^*}(x) \tag{19}$$

and the optimal continuation set D^*. As before, we can show that the optimal policy is a threshold rule: D^* is an open interval. In order to find D^*, we solve a stochastic differential equation and use the smooth-pasting principle (Oksendal 2003, p. 225): $dR_{D^*}(x)/dx$ is continuous.

Proposition 6: For $g > 0$, the optimal continuation set which maximizes the objective function in Eq. (19) is $D^* = (\xi_E, \xi_I)$, where $-\infty < \xi_E < x^+ < \xi_I < \infty$; it is optimal to exit when $x \le \xi_E$, invest when $x \ge \xi_I$, and continue operations otherwise.

Proof. In Appendix C, we prove $D^* = (\xi_E, \xi_I)$ where $\xi_E < x^+ < \xi_I$. By Proposition 5, it is optimal to invest for *some* value of X_t, so we have $\xi_I < \infty$. Now we only need to prove $\xi_E > -\infty$.

Suppose $\xi_E = -\infty$ so that $D^* = (-\infty, \xi_I)$, and let x be less than $\min\{\xi_I, \xi_1 - b\}$. Then

$$\begin{aligned} V_1(x) =& E^x[\int_0^{\tau_{D^*}} X_t e^{-\alpha t} dt + e^{-\alpha \tau_{D^*}} h(\xi_I)] \\ =& x/\alpha + \mu/\alpha^2 - E^x[e^{-\alpha\tau_{D^*}}](x/\alpha + \mu/\alpha^2 - h(\xi_I)) - E^x[\tau_{D^*} e^{-\alpha\tau_{D^*}}]\mu/\alpha\,. \end{aligned}$$

Using the same argument used in the proof (ii) of Proposition 5, it is easy to verify that $E^x[e^{-\alpha\tau_{D^*}}] = \exp[\gamma_p(x-\xi_I)]$ and $E^x[\tau_{D^*} e^{-\alpha\tau_{D^*}}] = -\partial_\alpha \gamma_p e^{\gamma_p(x-\xi_I)}(x-\xi_I)$ for $x < \xi_I$. Thus, for sufficiently large values of $|x|$ when $x < \xi_I$, $V_1(x) < 0$, contradicting the assumption $\xi_E = -\infty$. ■

If the profit rate is x^+, then there is positive probability that the profit rate will increase to a value bigger than x^+ in the immediate future. Hence, the expected return from waiting is positive, so $V_1(x^+) > 0$ and $x^+ \in D^* = (\xi_E, \xi_I)$. By Proposition 6 , the firm's optimal policy is to stop whenever $X_t \notin (\xi_E, \xi_I)$ and receive the reward $h(X_t)$. Notice that $V_0^+(\xi_I + b) - k > 0$ and $V_0^+(\xi_E + b) - k < 0$ because $\xi_E < x^+ < \xi_I$. Therefore, the firm's optimal action at the stopping time τ_{D^*} depends on which end of the interval (ξ_E, ξ_I) X_t hits first. It is optimal to exit if X_t hits ξ_E at time τ_{D^*}, and it is optimal to invest if X_t hits ξ_I at time τ_{D^*}.

We are now ready to construct the optimal return function and the equations for the thresholds. Because D^* is a bounded interval, by Appendix A, the solution $V_1(x)$ satisfies the PDE

$$(\partial_t + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2)[e^{-\alpha t}V_1(x)] = -xe^{-\alpha t} \quad \text{for all } x \in D^* \tag{20}$$

and $V_1(x) = h(x)$ for $x \notin D^*$. Given the set D^*, there is a unique solution to the PDE and the boundary conditions on $\{\xi_E, \xi_I\}$. In addition, by Theorem A of Appendix A, $V_1(\cdot)$ must satisfy the smooth-pasting conditions $\partial_x V_1(x) = \partial_x h(x)$ on the boundary $\{\xi_E, \xi_I\}$. The solution to the PDE and the boundary conditions is

$$V_1(x) = \begin{cases} x/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p x} + a_2 e^{\gamma_n x}, & \text{for } x \in D^* = (\xi_E, \xi_I) \\ h(x) \quad , & \text{otherwise} \end{cases} \tag{21}$$

where γ_p and γ_n are defined in Eq. (8). The unknown parameters, a_1, a_2, ξ_E, and ξ_I, are determined by the boundary conditions $V_1(x) = h(x)$ and the smooth-pasting conditions $\partial_x V_1(x) = \partial_x h(x)$ on the boundary $\{\xi_E, \xi_I\}$ of D^*. See Eqs. (41) to (44) in Appendix D for details.

5.3 Comparative Statics

In this section, we effect a comparative statics analysis of $V_1(\cdot)$, ξ_E, and ξ_I. We first examine the comparative statics of $V_1(\cdot)$ with respect to μ and σ.

Proposition 7: For all $x \in \mathbb{R}$, $V_1(x)$ is non-decreasing in μ and σ. In particular, $V_1(x)$ is strictly increasing in μ for $x > \xi_E$.

Proof. To begin, note that $h(\cdot)$ is convex and non-decreasing because $V_0^+(\cdot)$ is convex and non-decreasing. Also note that $h(\cdot)$ is non-decreasing in both μ and σ because $V_0(\cdot)$ is non-decreasing in μ and σ as shown at the end of Sec. 4.

To show that $V_1(\cdot)$ is non-decreasing in μ, we employ the argument used in Eq. (13). Let $V_1(x;\mu)$ and $h(x;\mu)$ denote the dependence of $V_1(x)$ and $h(x)$ on the initial (pre-investment) drift μ. Then for any $\beta > 0$ and $x > \xi_E$,

$$\begin{aligned} V_1(x;\mu) &= E^x[\int_0^{T_\mu} X_t e^{-\alpha t} dt + e^{-\alpha T_\mu} h(X_{T_\mu};\mu)] \\ &< E^x[\int_0^{T_\mu} (X_t + \beta t) e^{-\alpha t} dt + e^{-\alpha T_\mu} h(X_{T_\mu} + \beta T_\mu; \mu + \beta)] \le V_1(x;\mu+\beta) \end{aligned}$$

where T_μ is the optimal stopping time which maximizes $R_D(x)$ when the drift is μ. In establishing the strict inequality, we used the fact that $T_\mu > 0$ for $x > \xi_E$, $h(x;\mu)$ is non-decreasing in x and μ, and T_μ is suboptimal when the drift is $\mu + \beta$.

Similarly, define $V_1(x;\sigma^2)$ and $h(x;\sigma^2)$ as $V_1(x)$ and $h(x)$ with volatility σ and let $\bar{B}_t$ denote a one-dimensional standard Brownian motion which is independent of B_t. Let T_{σ^2} denote the optimal stopping time which maximizes $R_D(x)$ when the volatility is σ. Because T_{σ^2} is a stopping time for the process $X_t = x + \mu t + \sigma B_t$, it is independent of $\bar{B}_t$ so that $E^x[\int_0^{T_{\sigma^2}} \bar{B}_t e^{-\alpha t} dt] = 0$. Then

$$\begin{aligned}
V_1(x;\sigma^2) =& E^x[\int_0^{T_{\sigma^2}} X_t e^{-\alpha t} dt + e^{-\alpha T_{\sigma^2}} h(X_{T_{\sigma^2}};\sigma^2)] \\
=& E^x[\int_0^{T_{\sigma^2}} (X_t + \sigma_1 \bar{B}_t) e^{-\alpha t} dt + e^{-\alpha T_{\sigma^2}} h(X_{T_{\sigma^2}};\sigma^2)] \\
\le& E^x[\int_0^{T_{\sigma^2}} (X_t + \sigma_1 \bar{B}_t) e^{-\alpha t} dt + e^{-\alpha T_{\sigma^2}} h(X_{T_{\sigma^2}} + \sigma_1 \bar{B}_t;\sigma^2)] \\
\le& E^x[\int_0^{T_{\sigma^2}} (X_t + \sigma_1 \bar{B}_t) e^{-\alpha t} dt + e^{-\alpha T_{\sigma^2}} h(X_{T_{\sigma^2}} + \sigma_1 \bar{B}_t;\sigma^2 + \sigma_1^2)] \\
\le& V_1(x;\sigma^2 + \sigma_1^2) \ ,
\end{aligned}$$

where the first inequality follows from Jensen's inequality $E^x[h(X_{T_{\sigma^2}} + \sigma_1 \bar{B}_t;\sigma^2)] \ge E^x[h(X_{T_{\sigma^2}};\sigma^2)]$, the second inequality follows from $h(\cdot;\sigma^2)$ non-decreasing in σ^2, and the final inequality follows from the suboptimality of T_{σ^2} when the volatility is $\sigma^2 + \sigma_1^2$. ■

The comparative statics of ξ_E follows easily from Proposition 7.

Corollary 8: The exit threshold ξ_E satisfies $\partial_\mu \xi_E < 0$ and $\partial_{\sigma^2} \xi_E \le 0$.

Proof. Noting that $\xi_E = \inf\{x : V_1(x) > 0\}$, this result follows from the fact that $V_1(\cdot)$ is strictly increasing in μ for $x > \xi_E$ and non-decreasing σ. ■

In contrast, the comparative statics of ξ_I is considerably more complicated. Because $V_1(x) > V_0^+(x+b) - k$ if and only if $x < \xi_I$, $\xi_I = \sup\{x : V_1(x) - [V_0^+(x+b) - k] > 0\}$. Hence, the dependence of both $V_1(\cdot)$ and $V_0^+(\cdot)$ on μ and σ determine the comparative statics of ξ_I. In order to examine the comparative statics of ξ_I, we need to study the equations for both ξ_E and ξ_I. Equations (45) and (46) of Appendix D can be rewritten as

$$\begin{aligned}
\xi_E - \xi_0 &= e^{-\gamma_p(\xi_I - \xi_E)}[-g + (\lambda^{-1} - \gamma_n^{-1}) e^{\lambda(\xi_I + b - \xi_1)}] \\
&= e^{-\gamma_n(\xi_I - \xi_E)}[-g + (\gamma_p^{-1} - \gamma_n^{-1}) + (\gamma_n^{-1} - \gamma_p^{-1}) e^{\lambda(\xi_I + b - \xi_1)}] \ ,
\end{aligned}$$

where λ is given by (15). Note that a closed-form expression for ξ_I and ξ_E can not be obtained from the above equations.

Using the closed-form expression for ξ_0, it was straightforward to effect a complete comparative statics analysis of ξ_0. Lack of a closed-form expression impairs our ability to effect a comparative statics analysis of ξ_I. However, we can obtain useful insights by examining the leading-order terms of ξ_I in power series expansions of g when b is close to $\alpha k - \delta/\alpha$ (g is small) and when b is large (g is large). We do not consider δ large because we restrict our discussions to the interesting case $\mu^+ < 0$, i.e., the profit stream is in decline even after investment.

Using the expansions given in Propositions D1 and D2 of Appendix D, we obtain the limiting behavior of ξ_I and ξ_E. As $g \to 0$, we find $\xi_E \to \xi_0$ and $\xi_I \to \infty$; this echoes the intuition that it is almost never optimal to invest when g is close to zero. In the other limit where $b \to \infty$, we find $\xi_E \to -\infty$ and $\xi_I - \xi_E \to 0$; this occurs because it is optimal to invest whenever b is sufficiently large.

Proposition 9: For sufficiently small values of g, (i) $\partial_{\sigma^2}\xi_I > 0$ and $\partial_{\sigma^2}\xi_E < 0$; (ii) $\partial_\mu \xi_I < 0$.
Proof. Take the partial derivatives of Eqs. (47) and (48) with respect to μ and σ^2 and use Eqs. (9) and (10) to obtain the statements (i) and (ii). ■

Proposition 10: For sufficiently large b, (i) $\partial_{\sigma^2}\xi_I < 0$ and $\partial_{\sigma^2}\xi_E < 0$; (ii) $\partial_\mu \xi_I < 0$.

Proof. (i) From the definition of ξ_0 and Eq. (50), we have (a function $f(x)$ such that $f(x) \to 0$ as $x \to \infty$ is said to be $o(1)$)

$$\partial_{\sigma^2}\xi_E = -\gamma_n^{-2}\partial_{\sigma^2}\gamma_n + \partial_{\sigma^2}\theta + o(1) = -z(e^z - 1)^{-1}\lambda^{-2}\partial_{\sigma^2}\lambda + o(1)\,,$$

where $z \equiv -\lambda(\theta + \alpha k + \gamma_n^{-1} - \lambda^{-1}) > 0$, θ is defined by Eq. (52), and $\partial_{\sigma^2}\theta$ is given by Eq. (55). Note that z and θ are independent of b so that they are not affected when we take the limit $b \to \infty$. Because $\partial_{\sigma^2}\lambda > 0$ from Eq. (15), we have $\partial_{\sigma^2}\xi_E < 0$ for sufficiently large b. From Eq. (51), we have $\partial_{\sigma^2}(\xi_I - \xi_E) \to 0$ as $b \to \infty$ so that

$$\partial_{\sigma^2}\xi_I = \partial_{\sigma^2}\xi_E + \partial_{\sigma^2}(\xi_I - \xi_E) = -z(e^z - 1)^{-1}\lambda^{-2}\partial_{\sigma^2}\lambda + o(1)\,.$$

Thus, $\partial_{\sigma^2}\xi_I < 0$ for sufficiently large b.

(ii) By Corollary 8, $\partial_\mu \xi_E < 0$; however, it is instructive to show, from Eq. (51) and (56),

$$\partial_\mu \xi_E = \partial_\mu \xi_0 + \partial_\mu(\xi_E - \xi_0) = -\alpha^{-1} - z(e^z - 1)^{-1}\lambda^{-2}\partial_\mu \lambda_\mu + o(1)\,.$$

From Eq. (51), we have $\partial_\mu(\xi_I - \xi_E) \to 0$ as $b \to \infty$, so

$$\partial_\mu \xi_I = \partial_\mu \xi_E + \partial_\mu(\xi_I - \xi_E) = -\alpha^{-1} - z(e^z - 1)^{-1}\lambda^{-2}\partial_\mu \lambda_\mu + o(1)\,.$$

From the definition of λ in Eq. (15), it is easy to verify that $-\alpha^{-1} - z(e^z - 1)^{-1}\lambda^{-2}\partial_\mu\lambda_\mu < 0$ for any $z > 0$. Thus, $\partial_\mu \xi_I < 0$ for sufficiently large b. ■

When g is small, $\partial_{\sigma^2}\xi_E < 0$ and $\partial_{\sigma^2}\xi_I > 0$: as the uncertainty σ increases, it is optimal to wait longer to take advantage of the upturn potential before taking an irreversible action. This is similar to the numerical result obtained by Dixit (1989): the entry (exit) threshold increases (decreases) in the volatility. However, when g is large, Proposition 10 (i) asserts that $\partial_{\sigma^2}\xi_E < 0$ and $\partial_{\sigma^2}\xi_I < 0$. Notice that the result $\partial_{\sigma^2}\xi_I < 0$ stands in contrast to the conventional intuition inherited from real options theory. This counterintuitive result obtains because the return from investment, $V_0^+(x+b) - k$, depends on σ. It is worthy of note that the return from investment has dependence on σ only because exit is possible after investment.

6 Conclusions

Our analysis of investment under deteriorating conditions is congruent with empirical reality as exemplified by the hard disk drive industry and many other obsolescent technologies: it can be optimal to invest even in the face of a declining profit stream and eventual displacement from the market. On the other hand, it can be optimal to remain in the market even if the current profit rate is negative but above a threshold; it is optimal to exit only when the profit rate has deteriorated sufficiently. In particular, we obtain the closed-form solution for the post-investment threshold using stochastic calculus.

We also effect a comparative statics analysis of the optimal thresholds with respect to the volatility. As explained by Dixit (1992) and illustrated by McDonald and Siegel (1986) and Dixit (1989), the intuition inherited from real options theory suggests that it is optimal to delay an irreversible action longer as the degree of uncertainty increases. In the basic model of Sec. 4, for instance, the exit threshold ξ_0 always decreases in the volatility σ. Similarly, in the model of Sec. 5, the exit threshold ξ_E decreases in σ. The same intuition suggests that ξ_I increases in σ. Indeed, ξ_I increases in σ for sufficiently small g. However, we find that ξ_I decreases in σ for sufficiently large g: if the boost in the profit rate is sufficiently large, then it is optimal to invest earlier as the uncertainty about the future profit stream increases. This counterintuitive result is due to the firm's eventual exit, a salient feature of our model. The firm can take advantage of the volatility after investment if post-investment exit is possible, so an increase in volatility induces an increase in the expected return from investment and an increase in ξ_I for sufficiently large g.

Acknowledgments

I would like to thank Steven Lippman for very helpful suggestions and discussions. This research is partially supported by the Price Center for Entrepreneurial Studies at UCLA.

Appendix A: Return Function for Bounded Continuation Sets

In this section, we present two preliminaries to the proof of Propositions 1 and 6. First, we provide a method of constructing the discounted expected return R_D for the continuation set $D = (x_1, x_2)$

$$R_D(x) = E^x[\int_0^{\tau_D} e^{-\alpha st}\varphi(X_t)dt + e^{-\alpha\tau_D}h(X_{\tau_D})] \tag{22}$$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the time of the first exit from D and $X_t = X_0 + \mu t + \sigma B_t$. Here $h(\cdot)$ is the lump sum payoff upon exit from D. Second, we prove that provided that D^* is a bounded interval, the optimal return function has to satisfy the smooth-pasting condition $dR_{D^*}(x)/dx = dh(x)/dx$ for $x \in \{x_1, x_2\}$ in addition to the boundary conditions $R_{D^*}(x) = h(x)$ for$x \in \{x_1, x_2\}$. These results will be used to construct the optimal return functions given by Eqs. (5) and (21).

The infinitesimal generator $\mathcal{L}$ (Oksendal 2003, p. 121) is defined by

$$\mathcal{L}f(t,x) = \lim_{u\downarrow 0} \frac{E[f(t+u, X_{t+u})|X_t = x] - f(t,x)}{u} . \tag{23}$$

The generator $\mathcal{L}$ is well-defined for any twice continuously differentiable function $f(\cdot)$. Because $\{X_t\}$ is a Brownian motion with drift μ and volatility σ, the infinitesimal generator $\mathcal{L}$ of the process (t, X_t) is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} . \tag{24}$$

From the uniqueness theorems for Dirichlet-Poisson problems on Oksendal (2003) pp. 176-177, $R_D(x)$ satisfies

$$\mathcal{L}[e^{-\alpha t}R_D(x)] = -\varphi(x)e^{-\alpha t} \ \text{ for } x \in (x_1, x_2) . \tag{25}$$

The condition Eq. (25) is not necessarily guaranteed if (x_1, x_2) is unbounded, however. Define $R(x; x_1, x_2)$ by $R(x; x_1, x_2) \equiv R_D(x)$. From Eq. (22), the function $R(x; x_1, x_2)$ also satisfies the boundary conditions

$$R(x_1; x_1, x_2) = h(x_1) , \tag{26}$$

$$R(x_2; x_1, x_2) = h(x_2) . \tag{27}$$

We can attain intuitive understanding of Eq. (25). Applying Dynkin's formula (Oksendal 2003, p. 125; Harrison 1985, p. 74) to Eq. (23), we obtain

$$E^x[f(\tau, X_\tau)] = f(0,x) + E^x[\int_0^{\tau} \mathcal{L}f(t, X_t)dt] , \tag{28}$$

which is reminiscent of the fundamental theorem of calculus. Now we replace $f(t,x)$ with $e^{-\alpha t}R_D(x)$ and τ

with τ_D in Eq. (28) and use the fact that $R_D(X_{\tau_D}) = h(X_{\tau_D})$, and we obtain

$$R_D(x) = E^x[-\int_0^{\tau_D} \mathcal{L}[e^{-\alpha t} R_D(X_t)]dt + e^{-\alpha\tau_D} h(X_{\tau_D})] \ . \tag{29}$$

Identifying $\mathcal{L}[e^{-\alpha t} R_D(x)]$ with $-\varphi(x)e^{-\alpha t}$, Eqs. (22) and (29) coincide.

We now let $\varphi(x) = x$ as in our model in Sec. 4 and 5. The solution to the partial differential equation in Eq. (25) can be expressed as

$$R(x;x_1,x_2) = \alpha^{-1}(x+\mu/\alpha) + a_1(x_1,x_2)e^{\gamma_p x} + a_2(x_1,x_2)e^{\gamma_n x}\,, \tag{30}$$

where γ_p and γ_n are defined by Eq. (8). (See also Harrison 1985, Chapter 3.) The coefficients $a_i(x_1,x_2)$ $(i = 1,2)$ are determined by the boundary conditions in Eqs. (26) and (27):

$$a_1(x_1,x_2) = \frac{e^{\gamma_n x_2}[h(x_1) - \alpha^{-1}(x_1+\mu/\alpha)] - e^{\gamma_n x_1}[h(x_2) - \alpha^{-1}(x_2+\mu/\alpha)]}{e^{\gamma_p x_1+\gamma_n x_2} - e^{\gamma_p x_2+\gamma_n x_1}}\,, \tag{31}$$

$$a_2(x_1,x_2) = \frac{-e^{\gamma_p x_2}[h(x_1) - \alpha^{-1}(x_1+\mu/\alpha)] + e^{\gamma_p x_1}[h(x_2) - \alpha^{-1}(x_2+\mu/\alpha)]}{e^{\gamma_p x_1+\gamma_n x_2} - e^{\gamma_p x_2+\gamma_n x_1}}\,. \tag{32}$$

Next, we show that if $D = (x_1,x_2)$ is the optimal continuation set, then $R_D(x)$ satisfies the smooth-pasting condition: $dR_{D^*}(x)/dx = dh(x)/dx$ for $x \in \{x_1,x_2\}$.

Theorem A: Suppose that the optimal continuation set D which maximizes $R_D(x)$ defined in Eq. (22) is a bounded interval $D = (x_1,x_2)$ and that $h(\cdot)$ is continuously differentiable at x_1 and x_2. Then $R_D(x)$ satisfies the smooth-pasting condition: $R_D(\cdot)$ is differentiable at x_1 and x_2.

Proof. Because $R(x;x_1,x_2)$ is differentiable in x_1 and x_2, the necessary conditions for (x_1,x_2) to be the optimal continuation set are the first-order conditions:

$$\partial_{x_1} R(x;x_1,x_2) = 0 \ \text{ and } \quad \partial_{x_2} R(x;x_1,x_2) = 0 \ \text{ for all } \ x\,.$$

From the form of $R(x;x_1,x_2)$ in Eq. (30), the above conditions hold if and only if $\partial_{x_1} a_i(x_1,x_2) = 0$ and $\partial_{x_2} a_i(x_1,x_2) = 0$ for $i = 1$ and 2. Taking the total derivatives of Eqs. (26) and (27) with respect to x_1 and x_2, we have

$$\begin{aligned}
\frac{dR(x_1;x_1,x_2)}{dx_2} &= \partial_{x_2} a_1(x_1,x_2)e^{\gamma_p x_1} + \partial_{x_2} a_2(x_1,x_2)e^{\gamma_n x_1} = 0 \\
\frac{dR(x_2;x_1,x_2)}{dx_1} &= \partial_{x_1} a_1(x_1,x_2)e^{\gamma_p x_2} + \partial_{x_1} a_2(x_1,x_2)e^{\gamma_n x_2} = 0 \\
\frac{dR(x_1;x_1,x_2)}{dx_1} &= \lim_{x\downarrow x_1} \partial_x R(x;x_1,x_2) + \partial_{x_1} a_1(x_1,x_2)e^{\gamma_p x_1} + \partial_{x_1} a_2(x_1,x_2)e^{\gamma_n x_1} = \frac{dh(x)}{dx}|_{x=x_1}
\end{aligned}$$

$$\frac{dR(x_2;x_1,x_2)}{dx_2} = \lim_{x\uparrow x_2} \partial_x R(x;x_1,x_2) + \partial_{x_2} a_1(x_1,x_2) e^{\gamma_p x_2} + \partial_{x_2} a_2(x_1,x_2) e^{\gamma_n x_2} = \frac{dh(x)}{dx}|_{x=x_2} .$$

If we impose the necessary conditions $\partial_{x_j} a_i(x_1,x_2) = 0$ ($i = 1,2$ and $j = 1,2$) for optimality of x_1 and x_2, we obtain the smooth-pasting conditions: $dR_D(x)/dx = dh(x)/dx$ for $x = x_1$ and x_2 . ■

Appendix B: Return Function for Unbounded Continuation Sets

In this Appendix, we consider the continuation set $D = (\xi,\infty)$ and show that the function $R_D(x)$, defined in Eq. (3), satisfies the partial differential equation

$$\mathcal{L}[e^{-\alpha t} R_D(x)] = -x e^{-\alpha t} \text{ for } x \in D , \tag{33}$$

where $\mathcal{L}$ is defined by Eq. (24), along with the boundary condition $R_D(\xi) = 0$. The solution to this kind of partial differential equations exists and is well-known (Arfken 1985, Chapter 8).

In the following, we extend Eq. (25) to unbounded intervals such as (ξ,∞). We first consider a sequence of sets $D_n = (\xi,n)$ which converges to D and a sequence of functions

$$\phi_n(t,x) = e^{-\alpha t} E^x[\int_0^{T_n} e^{-\alpha u} X_u du] ,$$

where $T_n \equiv \tau_{D_n}$. Because each D_n is bounded, we can apply Eq. (25) from Appendix A. Hence, $\phi_n(\cdot,\cdot)$ satisfies $\mathcal{L}\phi_n(t,x) = -e^{-\alpha t} x$ for $x \in D_n$ with the boundary conditions $\phi_n(t,X_{T_n}) = 0$. The challenge is to establish Eq. (33) when D is unbounded.

We now employ the dominated convergence theorem to show that

$$\lim_{n\to\infty} \phi_n(t,x) = e^{-\alpha t} E^x[\int_0^{\tau_D} e^{-\alpha u} X_u du] . \tag{34}$$

Because

$$|\int_0^{T_n} e^{-\alpha u} X_u du| \le \int_0^{T_n} e^{-\alpha u} |X_u| du \le \int_0^{\infty} e^{-\alpha u} |X_u| du ,$$

we only need to show that $\int_0^\infty e^{-\alpha u}|X_u|du$ is E^x-integrable. From $X_t = x + \mu t + \sigma B_t$,

$$E^x[\int_0^\infty e^{-\alpha u}|X_u|du] \quad \le \quad E^x[\int_0^\infty e^{-\alpha u}(|\mu u| + |x| + |\sigma B_u|)du] .$$

From the fact that $|z| < 1 + z^2$ and that $E^x[\int_0^\infty B_u^2 du] = \int_0^\infty E^x[B_u^2]du$ from Fubini's theorem,

$$E^x[\int_0^\infty e^{-\alpha u}(|\sigma B_u|)du] \quad < \quad E^x[\int_0^\infty e^{-\alpha u}(1+\sigma^2 B_u^2)du] = \int_0^\infty e^{-\alpha u}(1+\sigma^2 E^x[B_u^2])du$$

$$= \int_0^\infty e^{-\alpha u}(1+\sigma^2 u)du = \alpha^{-1} + \sigma^2\alpha^{-2} < \infty\,.$$

We note that $\int_0^\infty e^{-\alpha u}(|\mu u| + |x|)du = |\mu|\alpha^{-2} + |x|\alpha^{-1}$. Hence, $E^x[\int_0^\infty e^{-\alpha u}|X_u|du] \leq (1+|x|)\alpha^{-1} + (|\mu| + \sigma^2)\alpha^{-2} < \infty$. Now we use the fact that

$$\lim_{n\to\infty}\int_0^{T_n} e^{-\alpha u}X_u du = \int_0^{\tau_D} e^{-\alpha u}X_u du\,,$$

the limit of which exists a.s. even on the set $\{\tau_D = \infty\}$ because of the law of iterated logarithm (p.66, Oksendal 2003) which constrains the magnitude of B_t in the large-t limit as follows:

$$\limsup_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad \text{a.s.} \tag{35}$$

Then we apply the dominated convergence theorem to arrive at Eq. (34). Lastly, we can easily verify that $\phi(t,x) \equiv \lim_{n\to\infty}\phi_n(t,x)$ satisfies Eq. (33) and the boundary condition $\phi(t,\xi) = 0$. ■

Appendix C: Proof of Proposition 6

Suppose that $g = b + \delta/\alpha - k\alpha > 0$. The goal of this Appendix is to show that the optimal continuation set is a bounded interval of the form $D^* = (\xi_E, \xi_I)$ which contains x^+ defined in Eq. (16). In the proof of Proposition 1, the objective function is of the form

$$R_D(x) = r_0(x) + E^x[\int_0^{\tau_D} f(t,X_t)dt]\,, \tag{36}$$

and it is easy to see that it is optimal to continue at $X_t = x$ when $f(t,x) > 0$. Hence, it is easy to identify at least a subset of the optimal continuation set. In Proposition 6, however, the objective function of Eq. (17) is not of the form in Eq. (36). In order to cast Eq. (17) into a form similar to Eq. (36), we first need to transform the objective function to one expressed without any integral with respect to time. To do so, we introduce the process W_t and the new lump sum payoff $\tilde{h}(t,x,w)$ as follows:

$$W_\tau \equiv \int_0^\tau e^{-\alpha t}X_t dt\,,$$
$$\tilde{h}(\tau, X_\tau, W_\tau) \equiv W_\tau + e^{-\alpha\tau}h(X_\tau)\,.$$

Then the objective function can be re-expressed as

$$V_1(x) = \sup_{\tau\in\mathcal{T}} E^x[\tilde{h}(\tau, X_\tau, W_\tau)]\,.$$

The infinitesimal generator $\mathcal{A}$ of the process (t, X_t, W_t) is given by

$$\mathcal{A} \equiv \partial_t + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2 + e^{-\alpha t} x \partial_w \, , \tag{37}$$

(see p.222, Oksendal 2003) and we have

$$\mathcal{A}\tilde{h}(t,x,w) = \begin{cases} (k\alpha - b - \delta/\alpha)e^{-\alpha t} + \alpha^{-1}\delta e^{-\alpha t + \lambda(x+b-\xi_1)} & \text{if } x > x^+ \, , \\ x e^{-\alpha t} & \text{if } x < x^+ \end{cases} . \tag{38}$$

Notice that $\mathcal{A}\tilde{h}(t,x,w)$ is not defined at $x = x^+$ because the function $\tilde{h}(t,x,w)$ is not differentiable at $x = x^+$. Given a bounded and connected continuation set $G \subset \mathbb{R}$ which does not contain x^+, the function $\tilde{h}(t,x,w)$ is twice continuously differentiable for all $x \in G$, so Dynkin's formula (Oksendal 2003, p. 125) applies:

$$E^x[\tilde{h}(\tau_G, X_{\tau_G}, W_{\tau_G})] = \tilde{h}(0,x,0) + E^x[\int_0^{\tau_G} \mathcal{A}\tilde{h}(t, X_t, W_t)dt] \, . \tag{39}$$

We denote $U \equiv \{x : \mathcal{A}\tilde{h}(t,x,w) > 0\}$, which is a bounded set because $\mathcal{A}\tilde{h}(t,x,w) < 0$ for sufficiently large $|x|$.

Claim 1: $U \subset D^*$.

Proof. Suppose $x \in U$ and $x > x^+$. We choose an interval $(x_1, x_2) \equiv H$ as the continuation set such that $x_2 > x_1 > x^+$ and $x \in H \subset U$. By Eq. (39),

$$E^x[\tilde{h}(\tau_H, X_{\tau_H}, W_{\tau_H})] = \tilde{h}(0,x,0) + E^x[\int_0^{\tau_H} \mathcal{A}\tilde{h}(t, X_t, W_t)dt] > \tilde{h}(0,x,0) \, .$$

Hence, x belongs to the optimal continuation set D^*. Similarly, we can show that if $x < x^+$ and $x \in U$ then $x \in D^*$. ■

Claim 2: The optimal continuation set D^* always contains a neighborhood of x^+.

Proof. Fix $\varepsilon > 0$ and set $F = (x^+ - \varepsilon, x^+ + \varepsilon)$. Select $x \in F$ and set $X_0 = x$. Define $f_\varepsilon(\cdot)$ by

$$f_\varepsilon(x) \equiv E^x[\int_0^{\tau_F} e^{-\alpha t} X_t dt + e^{-\alpha\tau_F} h(X_{\tau_F})] \, .$$

Define $o(\varepsilon)$ as a function of ε such that $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$. From the definition of $h(\cdot)$, $h(x^+ + \varepsilon) = C\varepsilon + o(\varepsilon)$ and $h(x^+ - \varepsilon) = 0$, where C is the positive constant given by $\lim_{x\downarrow x^+} \partial_x h(x)$. From Eqs. (30) – (32), we obtain $f_\varepsilon(x) = [x/\alpha + \mu/\alpha^2 + a_1(x^+ - \varepsilon, x^+ + \varepsilon)e^{\gamma_p x} + a_2(x^+ - \varepsilon, x^+ + \varepsilon)e^{\gamma_n x}]$. Expanding $a_i(x^+ - \varepsilon, x^+ + \varepsilon)$ for

$i=1,2$ in powers of ε, we have $f_\varepsilon(x^+) = [C\varepsilon/2 + o(\varepsilon)] > 0$ for sufficiently small ε, where C is the positive constant defined above. Hence, $x \in D^*$. Moreover, by continuity of $f_\varepsilon(\cdot)$, there is a neighborhood of x^+ within which the continuation policy $F = (x^+ - \varepsilon, x^+ + \varepsilon)$ gives a better return than does the non-continuation policy $F = \emptyset$. Therefore, the optimal continuation set always contains a non-empty neighborhood of x^+. ■

By Claim 2, there is an open neighborhood N of x^+ such that $x^+ \in N \subset D^*$. Then $\tilde{U} \equiv U \cup N$ is a single open interval because $\mathcal{A}\tilde{h}(t,x,w)$ decreases as x moves away from x^+ in each direction.

Claim 3: D^* does not have a subset D_d which is disconnected from $\tilde{U}$.

Proof. Suppose for the moment that D has a component D_d disconnected from $\tilde{U}$. If D_d is bounded, then we can use Dynkin's formula in Eq. (39) to show

$$E^x[\tilde{h}(\tau_{D_d}, X_{\tau_{D_d}}, W_{\tau_{D_d}})] = \tilde{h}(0,x,w) + E^x[\int_0^{\tau_{D_d}} \mathcal{A}\tilde{h}(s,X_t,W_t)dt] < \tilde{h}(0,x,w)\,, \tag{40}$$

for any $x \in D_d$. Equation (40) contradicts the assumption that $x \in D_d \subset D$. Hence, there is *no* bounded component D_d that is disconnected from $\tilde{U}$.

Even if D_d is unbounded, the same argument still applies. Defining $\tau_m = m \wedge \tau_{D_d}$ where m is a positive integer, we claim that $E^x[\tilde{h}(\tau_m, X_{\tau_m}, W_{\tau_m})] \to E^x[\tilde{h}(\tau_{D_d}, X_{\tau_{D_d}}, W_{\tau_{D_d}})]$ in the limit $m \to \infty$. We first note the law of iterated logarithm in Eq. (35), which implies that for any $\varepsilon > 0$, there is sufficiently large $T > 0$ such that $|B_t| < (1+\varepsilon)\sqrt{2t\log\log t}$ for all $t > T$ a.s. Hence, for sufficiently large t,

$$e^{-\alpha t}h(X_0 + \mu t - \sigma(1+\varepsilon)\sqrt{2t\log\log t}) < e^{-\alpha t}h(X_t) < e^{-\alpha t}h(X_0 + \mu t + \sigma(1+\varepsilon)\sqrt{2t\log\log t})$$

a.s. Moreover, for large enough t, $e^{-\alpha t}h(X_0 + \mu t + \sigma(1+\varepsilon)\sqrt{2t\log\log t})$ is an exponentially decreasing function of t for a fixed X_0. Therefore, there is some finite positive constant M such that

$$e^{-\alpha t}h(X_0 + \mu t + \sigma(1+\varepsilon)\sqrt{2t\log\log t}) < M$$

for all $t > 0$. Consequently, if m is sufficiently large and $\tau_m < \tau_{D_d}$, then

$$|e^{-\alpha\tau_{D_d}}h(X_{\tau_{D_d}}) - e^{-\alpha\tau_m}h(X_{\tau_m})| \le 2e^{-\alpha t}h(X_0 + \mu t + \sigma(1+\varepsilon)\sqrt{2t\log\log t}) < 2M$$

for all t, and $|e^{-\alpha\tau_{D_d}}h(X_{\tau_{D_d}}) - e^{-\alpha\tau_m}h(X_{\tau_m})| = 0$ if $\tau_m = \tau_{D_d}$. Hence, we can use the bounded convergence theorem:

$$\lim_{m\to\infty} E^x[e^{-\alpha\tau_{D_d}}h(X_{\tau_{D_d}}) - e^{-\alpha\tau_m}h(X_{\tau_m})] = E^x[\lim_{m\to\infty}(e^{-\alpha\tau_{D_d}}h(X_{\tau_{D_d}}) - e^{-\alpha\tau_m}h(X_{\tau_m}))] = 0\,.$$

In the proof of Proposition 1, we already showed that $\lim_{m\to\infty} E^x[\int_0^{\tau_m} e^{-\alpha s} X_s ds] = E^x[\int_0^{\tau_{D_d}} e^{-\alpha s} X_s ds]$. Therefore, the claim $E^x[\tilde{h}(\tau_m, X_{\tau_m}, W_{\tau_m})] \to E^x[\tilde{h}(\tau_{D_d}, X_{\tau_{D_d}}, W_{\tau_{D_d}})]$ is proved, and Eq. (40) still holds. ■

It follows from Claim 3 that D^* is an interval of the form $D^* = (\xi_E, \xi_I)$ where $\xi_E \leq x^+ \leq \xi_I$.

Appendix D: Equations for Thresholds

The coefficients a_1 and a_2 in Eq. (21) are determined by the boundary conditions

$$V_1(\xi_E) = \xi_E/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p \xi_E} + a_2 e^{\gamma_n \xi_E} = h(\xi_E) = 0\,, \tag{41}$$

$$\begin{aligned} V_1(\xi_I) &= \xi_I/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p \xi_I} + a_2 e^{\gamma_n \xi_I} \\ &= h(\xi_I) = (\xi_I + b)/\alpha + \mu^+/\alpha^2 - (\alpha\lambda)^{-1} e^{\lambda(\xi_I + b - \xi_1)} - k\,, \end{aligned} \tag{42}$$

and the smooth-pasting conditions

$$\partial_x V_1(\xi_E) = \alpha^{-1} + \gamma_p a_1 e^{\gamma_p \xi_E} + \gamma_n a_2 e^{\gamma_n \xi_E} = \partial_x h(\xi_E) = 0\,, \tag{43}$$

$$\partial_x V_1(\xi_I) = \alpha^{-1} + \gamma_p a_1 e^{\gamma_p \xi_I} + \gamma_n a_2 e^{\gamma_n \xi_I} = \partial_x h(\xi_I) = \alpha^{-1}[1 - e^{\lambda(\xi_I + b - \xi_1)}]\,. \tag{44}$$

For notational convenience, we define $\Delta_{IE} \equiv \xi_I - \xi_E$ and $\Delta_{E0} \equiv \xi_E - \xi_0$. We eliminate a_1 and a_2 from Eqs. (41) to (44), and we obtain

$$\Delta_{E0} = -g e^{-\gamma_p \Delta_{IE}} + (\lambda^{-1} - \gamma_n^{-1}) e^{\lambda(\Delta_{IE} + \Delta_{E0} + b + \xi_0 - \xi_1)} e^{-\gamma_p \Delta_{IE}} \tag{45}$$

$$= -g e^{-\gamma_n \Delta_{IE}} + (\gamma_p^{-1} - \gamma_n^{-1}) + (\gamma_n^{-1} - \gamma_p^{-1}) e^{\lambda(\Delta_{IE} + \Delta_{E0} + b + \xi_0 - \xi_1)} e^{-\gamma_n \Delta_{IE}}\,, \tag{46}$$

where g is defined by Eq. (14).

In order to keep track of leading-order terms of power expansions of g, we introduce a notation to denote the subleading order terms: we say that $f(x) = o(j(x))$ if $f(x)/j(x) \to 0$ as $x \to 0$, where $f(x)$ and $j(x)$ are functions of x.

Proposition D1: In the small-g limit,

$$\Delta_{E0} = -g^{1-\gamma_p/\gamma_n} C(\delta)(1 + o(1))\,, \tag{47}$$

$$\Delta_{IE} = -\gamma_n^{-1} \ln(g^{-1})(1 + o(1))\,, \tag{48}$$

where $C(\delta) = [(\gamma_p^{-1} - \gamma_n^{-1})]^{\gamma_p/\gamma_n}$ if $\delta > 0$ and $C(\delta) = [(\gamma_p^{-1} - \gamma_n^{-1})(1 - e^{\gamma_n b})]^{\gamma_p/\gamma_n}$ if $\delta = 0$.

Proof. First, we notice that if $g=0$, then $\Delta_{E0}=\xi_E-\xi_0=0$ and $\Delta_{IE}=\xi_I-\xi_E=\infty$. Hence, $\Delta_{E0}\to 0$ and $\Delta_{IE}\to\infty$ as $g\to 0$.

Suppose that $\delta>0$. It is easy to show that the first term of the right-hand-side (RHS) of Eq. (45) strictly dominates the second term so that

$$g^{-1}\exp[\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)]\to 0 \quad \text{as } g\to 0\,. \tag{49}$$

From Eq. (49), the leading order terms in RHS of Eq. (46) are contained in the first two terms: $-ge^{-\gamma_n\Delta_{IE}}+(\gamma_p^{-1}-\gamma_n^{-1})$ in agreement with Eq. (48). From the fact that $\lim_{g\to 0}\Delta_{E0}=0$, the only possible leading order term of Δ_{IE} is $\gamma_n^{-1}\ln[g(\gamma_p^{-1}-\gamma_n^{-1})^{-1}]$. The leading-order terms of $\Delta_{IE}=\gamma_n^{-1}\ln[g(\gamma_p^{-1}-\gamma_n^{-1})^{-1}]+o(1)$ is consistent with the condition in Eq. (49) because $\lambda/\gamma_n>1$. Finally, using the leading-order term of Δ_{IE} in Eq. (45), we obtain Eq. (47). We repeat the same procedure with $\delta=0$ to complete the statements of Proposition D1. ■

Similarly, we also say that $f(x)=o\,(j(x))$ if $f(x)/j(x)\to 0$ as $x\to\infty$.

Proposition D2: In the large-b limit,

$$\Delta_{E0} = -g+\theta+o(1) \tag{50}$$

$$\Delta_{IE} = -g^{-1}(\gamma_p\gamma_n)^{-1}(1-\lambda\theta-\lambda/\gamma_n)+o(g^{-1}) \tag{51}$$

where θ is the unique positive solution to the equation

$$\theta=-\gamma_n^{-1}+\lambda^{-1}e^{\lambda(\theta+\alpha k-\delta/\alpha+\xi_0-\xi_1)}\,. \tag{52}$$

Proof. In the limit $b\to\infty$, we can show that $\Delta_{IE}=\xi_I-\xi_E\to 0$ and $\Delta_{E0}=\xi_E-\xi_0\to-\infty$ are the only correct asymptotic behaviors. We notice that a necessary condition for the firm at time t to have non-negative return from investment is that the boosted profit rate X_t+b exceeds ξ_1, so $\xi_I+b>\xi_1$ must be satisfied. Hence, in the limit $b\to\infty$, $e^{\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)}$ is bounded by 1 because
$\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1=\xi_I+b-\xi_1>0$ and $\lambda<0$.

From RHS of Eq. (45), the leading-order term of Δ_{E0} is $-g$. We claim that the second-leading-order term of Δ_{E0} is a positive constant, independent of g. Suppose that the second-order term of Δ_{E0} grows in g, but does so more slowly than g. Then the first and second leading-order terms of Eqs. (45) and (46) are $-ge^{-\gamma_p\Delta_{IE}}=$

$-g+g\gamma_p\Delta_{IE}(1+o(1))$ and $-ge^{-\gamma_n\Delta_{IE}}=-g+g\gamma_n\Delta_{IE}(1+o(1))$ respectively, which are inconsistent because $\gamma_p\neq\gamma_n$. Thus, the second leading order term of Δ_{E0} is a constant independent of g. Hence, we can express Δ_{E0} as in Eq. (50) where θ is a constant yet to be determined. Then Eqs. (45) and (46) can be re-expressed as

$$\Delta_{E0} = -g+g\gamma_p\Delta_{IE}+(\lambda^{-1}-\gamma_n^{-1})e^{\lambda(\theta-\delta/\alpha+k\alpha+\xi_0-\xi_1)}+o(1)\,, \tag{53}$$

$$\Delta_{E0} = -g+g\gamma_n\Delta_{IE}+(\gamma_p^{-1}-\gamma_n^{-1})+(\lambda^{-1}-\gamma_p^{-1})e^{\lambda(\theta-\delta/\alpha+k\alpha+\xi_0-\xi_1)}+o(1)\,. \tag{54}$$

Thus, the leading-order term of Δ_{IE} converges to zero at least as fast as g^{-1} because otherwise Δ_{E0} has a second leading order term growing in g. Let us set $\Delta_{IE}=C/g+o(g^{-1})$ for some constant C. From Eqs. (50), (53) and (54), we arrive at $C=-(\gamma_p\gamma_n)^{-1}(1-\lambda\theta-\lambda/\gamma_n)$ where θ satisfies Eq. (52). ■

We need to obtain the comparative statics of θ in order to examine the comparative statics of ξ_I and ξ_E in the large-b limit in Sec. 5.3. From Eq. (52) and the implicit function theorem, the partial derivatives of θ with respect to σ^2 and μ are given by

$$\partial_{\sigma^2}\theta=\gamma_n^{-2}\partial_{\sigma^2}\gamma_n+\frac{e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}{1-e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})\lambda^{-1}\partial_{\sigma^2}\lambda\,, \tag{55}$$

$$\partial_{\mu}\theta=\gamma_n^{-2}\partial_{\mu}\gamma_n+\frac{e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}{1-e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})\lambda^{-1}\partial_{\mu}\lambda\,. \tag{56}$$

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